§ 8.2 Homogeneous Linear Systems

Introduction: In section 8.1 we saw that the general solution of the homogeneous system

$$\mathbf{X}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\mathbf{X} \text{ is } \quad \mathbf{X} = c_1\mathbf{X}_1 + c_2\mathbf{X}_2 = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix}e^{3t} + c_2\begin{pmatrix} 1 \\ -2 \end{pmatrix}e^{-t}$$

The solution vectors $\mathbf{X}_1$ and $\mathbf{X}_2$ have the form $\mathbf{X}_i = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}e^{\lambda_i t}$ where $k_1, k_2, \lambda_1,$ and λ_2 are constants.

Can we always find a solution of the form $\mathbf{X} = \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix}e^{\lambda t} = \mathbf{K}e^{\lambda t}$ for the general homogeneous linear first-order system $\mathbf{X}' = \mathbf{AX}$, where **A** is an $n \times n$ matrix of constants?

Eigenvalues and Eigenvectors: If $\mathbf{X} = \mathbf{K}e^{\lambda t}$ is a solution vector of the homogeneous linear system $\mathbf{X}' = \mathbf{AX}$, then $\mathbf{X}' =$ __________ , so the system becomes

$$\mathbf{K}\lambda e^{\lambda t} = \mathbf{AK}e^{\lambda t}$$

where **I** is the $n \times n$ identity matrix. The last matrix equation is equivalent to

$$\begin{aligned}
&(a_{11} - \lambda)k_1 + a_{12}k_2 + \ldots + a_{1n}k_n = 0 \\
&a_{21}k_1 + (a_{22} - \lambda)k_2 + \ldots + a_{2n}k_n = 0 \\
&\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
&a_{n1}k_1 + a_{n2}k_2 + \ldots + (a_{nn} - \lambda)k_n = 0
\end{aligned}$$

An obvious solution of this system is $k_1 = 0, k_2 = 0, \ldots, k_n = 0$ but we are looking for a nontrivial solution.

Fact: A homogeneous system of n linear equations in n unknowns has a nontrivial solution if and only if the determinant of the coefficient matrix is equal to 0.

So to find a nonzero solution **K** for $(\mathbf{A} - \lambda\mathbf{I})\mathbf{K} = \mathbf{0}$, we must have ________________________

This results in an nth degree polynomial in λ called the **characteristic equation** of the matrix **A**. The **eigenvalues** of **A** are the roots of the characteristic equation. A solution $\mathbf{K} \neq \mathbf{0}$ of $(\mathbf{A} - \lambda\mathbf{I})\mathbf{K} = \mathbf{0}$ corresponding to an eigenvalue λ is called an **eigenvector** of **A**. A solution of the homogeneous system is then $\mathbf{X} = \mathbf{K}e^{\lambda t}$.

Distinct Real Eigenvalues:

Theorem 8.2.1: General Solution – Homogeneous Systems
Let $\lambda_1, \lambda_2, ..., \lambda_n$ be n distinct real eigenvalues of the coefficient matrix $\mathbf{A}$ of the homogeneous system $\mathbf{X}' = \mathbf{AX}$ and let $\mathbf{K}_1, \mathbf{K}_2, ..., \mathbf{K}_n$ be the corresponding eigenvectors. Then the **general solution** of $\mathbf{X}' = \mathbf{AX}$ on the interval $(-\infty, \infty)$ is given by

$$\mathbf{X} = c_1\mathbf{K}_1 e^{\lambda_1 t} + c_2\mathbf{K}_2 e^{\lambda_2 t} + ... + c_n\mathbf{K}_n e^{\lambda_n t}$$

Example: Find the general solution of the system.

$$\frac{dx}{dt} = -5x + y$$
$$\frac{dy}{dt} = 4x - 2y$$

Example: Solve the IVP.

$$\mathbf{X}' = \begin{pmatrix} 1 & 1 & 4 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}\mathbf{X}, \qquad \mathbf{X}(0) = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$$

Repeated Eigenvalues: If m is a positive integer and $\left(\lambda - \lambda_1\right)^m$ is a factor of the characteristic equation, then λ_1 is an **eigenvalue of multiplicity *m*.**

- For some $n \times n$ matrices **A** it may be possible to find m linearly independent eigenvectors $\mathbf{K}_1, \mathbf{K}_2, ..., \mathbf{K}_m$ corresponding to an eigenvalue λ_1 of multiplicity m.

Eigenvalue of Multiplicity Two:

Example: Find the general solution of the system.

$$\frac{dx}{dt} = 3x + 2y + 4z$$
$$\frac{dy}{dt} = 2x + 2z$$
$$\frac{dz}{dt} = 4x + 2y + 3z$$

Note: The matrix in the previous example is a special kind of matrix called a **symmetric** matrix. An $n \times n$ matrix **A** is **symmetric** if its transpose $\mathbf{A}^T$ (where the rows and columns are interchanged) is the same as **A** – that is, $\mathbf{A}^T = \mathbf{A}$. It can be proved that if the matrix **A** in the system $\mathbf{X}' = \mathbf{A}\mathbf{X}$ is symmetric and has real entries, then we can always find n linearly independent eigenvectors $\mathbf{K}_1, \mathbf{K}_2, ..., \mathbf{K}_n$.

Second Solution: Now suppose that λ_1 is an eigenvalue of multiplicity two and that there is only one eigenvector associated with this value. A second solution will be of the form

$$\mathbf{X}_2 = \mathbf{K}te^{\lambda_1 t} + \mathbf{P}e^{\lambda_1 t}$$

where $\mathbf{K} = \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix}$ and $\mathbf{P} = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{pmatrix}$.

Let's plug this into the system $\mathbf{X}' = \mathbf{AX}$ and see what we get:

Example: Find the general solution of the system.

$$\frac{dx}{dt} = 7x + y$$
$$\frac{dy}{dt} = -4x + 3y$$

Eigenvalue of Multiplicity Three: If only one eigenvector is associated with an eigenvalue λ_1 of multiplicity three, we can find a second solution (as above) and a third solution of the form:

$$\mathbf{X}_3 = \mathbf{K}\frac{t^2}{2}e^{\lambda_1 t} + \mathbf{P}te^{\lambda_1 t} + \mathbf{Q}e^{\lambda_1 t}$$

where $\mathbf{K} = \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix}$, $\mathbf{P} = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{pmatrix}$, and $\mathbf{Q} = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix}$.

If we plug this into the system $\mathbf{X}' = \mathbf{AX}$ we will get:

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{K} = \mathbf{0}$$
$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{P} = \mathbf{K}$$
$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{Q} = \mathbf{P}$$

Example: Find the general solution of the system. $\mathbf{X}' = \begin{pmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{pmatrix}\mathbf{X}$

Complex Eigenvalues: We now look at solutions to the system $\mathbf{X}' = \mathbf{AX}$ where the eigenvalues of the matrix **A** are complex. With complex eigenvalues we are going to have the same problem that we had back when we were looking at second order differential equations. We want our solutions to only have real numbers in them so getting rid of the complex numbers here will be similar to how we did it back in section 4.3.

The characteristic equation of the system

$$\frac{dx}{dt} = 3x - 13y$$
$$\frac{dy}{dt} = 5x + y$$

is $\det(\mathbf{A} - \lambda\mathbf{I}) =$

From the quadratic formula we find the eigenvalues:

For $\lambda_1 = 2 + 8i$ we must solve:

Similarly, for $\lambda_2 = 2 - 8i$ we find:

We can verify that these solution vectors are linearly independent, so the general solution is:

Theorem 8.2.2: Solutions Corresponding to a Complex Eigenvalue
Let $\mathbf{A}$ be the coefficient matrix having real entries of the homogeneous system $\mathbf{X}' = \mathbf{A}\mathbf{X}$, and let $\mathbf{K}_1$ be an eigenvector corresponding to the complex eigenvalue $\lambda_1 = \alpha + i\beta$. Then

$$\mathbf{K}_1 e^{\lambda_1 t} \quad \text{and} \quad \bar{\mathbf{K}}_1 e^{\bar{\lambda}_1 t}$$

are solutions of $\mathbf{X}' = \mathbf{A}\mathbf{X}$.

We would like to rewrite our solution in terms of real numbers. Use Euler's formula to write

$$e^{(2+8i)t} =$$

$$e^{(2-8i)t} =$$

Then, after we multiply complex numbers, collect terms, and replace $c_1 + c_2$ by C_1 and $(c_1 - c_2)i$ by C_2, the general solution becomes

Theorem 8.2.3: Real Solutions Corresponding to a Complex Eigenvalue
Let $\lambda_1 = \alpha + i\beta$ be a complex eigenvalue of the coefficient matrix $\mathbf{A}$ in the homogeneous system $\mathbf{X}' = \mathbf{A}\mathbf{X}$. Then

$$\mathbf{X}_1 = \left(\mathbf{B}_1 \cos\beta t - \mathbf{B}_2 \sin\beta t\right) e^{\alpha t}$$
$$\mathbf{X}_2 = \left(\mathbf{B}_2 \cos\beta t + \mathbf{B}_1 \sin\beta t\right) e^{\alpha t}$$

are linearly independent solutions of $\mathbf{X}' = \mathbf{A}\mathbf{X}$ on $(-\infty, \infty)$.

The matrices $\mathbf{B}_1$ and $\mathbf{B}_2$ are often denoted by $\mathbf{B}_1 = \operatorname{Re}(\mathbf{K}_1)$ and $\mathbf{B}_2 = \operatorname{Im}(\mathbf{K}_1)$.

Example: Find the general solution of the system.

$$\mathbf{X}' = \begin{pmatrix} 4 & 0 & 1 \\ 0 & 6 & 0 \\ -4 & 0 & 4 \end{pmatrix} \mathbf{X}$$