§ 8.1 Preliminary Theory – Linear Systems

Introduction: In this chapter we will be solving systems of linear first-order DEs. We are going to develop a general theory for these kinds of systems and a method of solution that utilizes some basic concepts from the algebra of matrices. Matrix notation and properties are used throughout this chapter so please review Appendix II or a linear algebra text if you are unfamiliar with these concepts.

Linear Systems: We are going to be looking at first-order, linear systems of differential equations. These terms mean the same thing that they have meant up to this point. The largest derivative anywhere in the system will be a first derivative and all unknown functions and their derivatives will only occur to the first power and will not be multiplied by other unknown functions.

$$\begin{aligned}
\frac{dx_1}{dt} &= a_{11}(t)x_1 + a_{12}(t)x_2 + \ldots + a_{1n}(t)x_n + f_1(t) \\
\frac{dx_2}{dt} &= a_{21}(t)x_1 + a_{22}(t)x_2 + \ldots + a_{2n}(t)x_n + f_2(t) \\
\vdots \quad & \qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
\frac{dx_n}{dt} &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \ldots + a_{nn}(t)x_n + f_n(t)
\end{aligned}$$

We assume that the coefficients a_{ij} as well as the functions f_i are continuous on a common interval *I*. When $f_i(t) = 0$, $i = 1, 2, \ldots, n$, the linear system above is said to be **homogeneous**: otherwise, it is **nonhomogeneous**.

Matrix Form of a Linear System: If **X**, **A**(*t*), and **F**(*t*) are defined as

$$\mathbf{X} = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \qquad \mathbf{A}(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \ldots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix}, \qquad \mathbf{F}(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix}$$

then the system of linear first-order DEs above can be written as

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \ldots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix}$$

or simply as ____________________ . (1)

If the system is homogeneous, its matrix form is then ____________________ . (2)

Example: Write the linear system in matrix form.

$$\frac{dx}{dt} = -3x + 4y + e^{-t}\sin(2t)$$
$$\frac{dy}{dt} = 5x + 9z + 4e^{-t}\cos(2t)$$
$$\frac{dz}{dt} = y + 6z - e^{-t}$$

Definition 8.1.1: Solution Vector
A **solution vector** on an interval I is any column matrix

$$\mathbf{X} = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

whose entries are differentiable functions satisfying the system (1) on the interval.

Example: Verify that the vector **X** is a solution of the given system.

$$\mathbf{X}' = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}\mathbf{X}; \quad \mathbf{X} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{t} + \begin{pmatrix} 4 \\ -4 \end{pmatrix} te^{t}$$

Initial Value Problem: Let t_0 denote a point on an interval *I* and

$$\mathbf{X}(t_0)=\begin{pmatrix} x_1(t_0) \\ x_2(t_0) \\ \vdots \\ x_n(t_0) \end{pmatrix} \quad \text{and} \quad \mathbf{X}_0=\begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{pmatrix}$$

where the γ_i, $i = 1, 2, \ldots, n$ are given constants. Then the problem

Solve: $\mathbf{X}' = \mathbf{A}(t)\mathbf{X} + \mathbf{F}(t)$
Subject to: $\mathbf{X}(t_0) = \mathbf{X}_0$

is an **initial value problem** on the interval.

Theorem 8.1.1: Existence of a Unique Solution
Let the entries of the matrices $\mathbf{A}(t)$ and $\mathbf{F}(t)$ be functions continuous on a common interval *I* that contains the point t_0. Then there exists a unique solution of the IVP on the interval.

Theorem 8.1.2: Superposition Principle
Let $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_k$ be a set of solution vectors of the homogeneous system (2) on an interval *I*. Then the linear combination

$$\mathbf{X} = c_1\mathbf{X}_1 + c_2\mathbf{X}_2 + \ldots + c_k\mathbf{X}_k$$

where the c_i, $i = 1, 2, \ldots, k$ are arbitrary constants is also a solution on the interval.

It can be verified that $\mathbf{X}_1 = \begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix}$ and $\mathbf{X}_2 = \begin{pmatrix} e^{-t} \\ -2e^{-t} \end{pmatrix}$ are solutions of the system

$\mathbf{X}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\mathbf{X}$. Then by the superposition principle **X** = ____________________________________
is another solution of the system.

Definition 8.1.2: Linear Dependence/Independence
Let $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_k$ be a set of solution vectors of the homogeneous system (2) on an interval *I*. We say that the set is **linearly dependent** on the interval if there exist constants $c_1, c_2, \ldots, c_k$, not all zero, such that

$$c_1\mathbf{X}_1 + c_2\mathbf{X}_2 + \ldots + c_k\mathbf{X}_k = \mathbf{0}$$

for every *t* in the interval. If the set of vectors is not linearly dependent on the interval, it is said to be **linearly independent**.

Theorem 8.1.3: Criterion for Linearly Independent Solutions

Let $\mathbf{X}_1 = \begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{pmatrix}$, $\mathbf{X}_2 = \begin{pmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{n2} \end{pmatrix}$, … , $\mathbf{X}_n = \begin{pmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{nn} \end{pmatrix}$

be *n* solution vectors of the homogeneous system (2) on an interval *I*. Then the set of solution vectors is linearly independent on *I* if and only if the **Wronskian**

$$W\left(\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_n\right) = \begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{vmatrix} \neq 0$$

for every *t* in the interval.

Note: Here, the definition of the Wronskian does not involve differentiation.

Definition 8.1.3: Fundamental Set of Solutions
Any set $\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_n$ of *n* linearly independent solution vectors of the homogeneous system (2) on an interval *I* is said to be a **fundamental set of solutions** on the interval.

Theorem 8.1.4: Existence of a Fundamental Set
There exists a fundamental set of solutions for the homogeneous system (2) on an interval *I*.

Theorem 8.1.5: General Solution – Homogeneous Systems
Let $\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_n$ be a fundamental set of solutions of the homogeneous system (2) on an interval *I*. Then the **general solution** of the system on the interval is

$$\mathbf{X} = c_1\mathbf{X}_1 + c_2\mathbf{X}_2 + ... + c_n\mathbf{X}_n$$

where the c_i, $i = 1, 2, \ldots, n$ are arbitrary constants.

Example: The vectors $\mathbf{X}_1 = \begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix}$ and $\mathbf{X}_2 = \begin{pmatrix} e^{-t} \\ -2e^{-t} \end{pmatrix}$ are solutions of the system $\mathbf{X}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\mathbf{X}$. Determine whether the vectors form a fundamental set on the interval $\left(-\infty, \infty\right)$.

Thus, ____________________________________ is the general solution of the system on the interval.

Nonhomogeneous Systems: A **particular solution** $\mathbf{X}_p$ on an interval *I* is any vector, free of arbitrary parameters, whose entries are functions that satisfy the system (1).

Theorem 8.1.6: General Solution – Nonhomogeneous Systems
Let $\mathbf{X}_p$ be a given solution of the nonhomogeneous system (1) on *I* and let

$$\mathbf{X} = c_1\mathbf{X}_1 + c_2\mathbf{X}_2 + \ldots + c_n\mathbf{X}_n$$

denote the general solution on the same interval of the associated homogeneous system (2). Then the **general solution** of the nonhomogeneous system on the interval is

$$\mathbf{X} = \mathbf{X}_c + \mathbf{X}_p$$

The general solution $\mathbf{X}_c$ of the associated homogeneous system (2) is called the **complimentary function** of the nonhomogeneous system (1).

Example: Verify the vector $\mathbf{X}_p = \begin{pmatrix} -2t - \frac{4}{7} \\ 6t + \frac{10}{7} \end{pmatrix}$ is a particular solution of the nonhomogeneous system

$\mathbf{X}' = \begin{pmatrix} 6 & 1 \\ 4 & 3 \end{pmatrix}\mathbf{X} + \begin{pmatrix} 6t \\ -10t + 4 \end{pmatrix}$ on $(-\infty, \infty)$.

The complimentary function on the same interval is $\mathbf{X}_c = c_1\begin{pmatrix} 1 \\ -4 \end{pmatrix}e^{2t} + c_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{7t}$.

Thus, ___________________________________ is the general solution of the system on the interval.