§ 7.3 Operational Properties

Introduction: We wish to increase our ability to find Laplace transforms of given functions easily. We don't always want to return to the definition of $\mathcal{L}\{f(t)\}$ each time we wish to calculate it. For example, $\mathcal{L}\{t^3\cos(4t)\}$ would be possible but laborious, $\mathcal{L}\{te^{2t}\cos(3t)\}$ even worse. We will learn techniques to calculate them rather quickly with the help of our table of transforms.

Translation on the *s*-axis:

Suppose we know $\mathcal{L}\{f(t)\} = F(s)$. We will find that calculating $\mathcal{L}\{e^{at}f(t)\}$ is easy. For example, consider $\mathcal{L}\{e^{6t}t^4\}$. By definition of the Laplace transform:

$$\mathcal{L}\{e^{6t}t^4\} = \int_0^\infty e^{-st}e^{6t}t^4\,dt = \int_0^\infty e^{-st+6t}t^4\,dt = \int_0^\infty e^{-(s-6)t}t^4\,dt$$

Similarly, suppose we know $\mathcal{L}\{f(t)\} = F(s)$, then

$$\mathcal{L}\{e^{at}f(t)\} = \int_0^\infty e^{-st}e^{at}f(t)\,dt = \int_0^\infty e^{-(s-a)t}f(t)\,dt = F(s-a)$$

which leads us to…

Theorem 7.3.1: The First Translation Theorem
If $\mathcal{L}\{f(t)\} = F(s)$ and a is any real number, then $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$.

We write: $\mathcal{L}\{e^{at}f(t)\} = \mathcal{L}\{f(t)\}\Big|_{s\to s-a}$

Example: Use the Translation Theorem to find the following:

a) $\mathcal{L}\{e^{7t}t^{10}\}$

b) $\mathcal{L}\{e^{-3t}\cos(2t)\}$

Inverse Form of Theorem 7.3.1:

Part b) of the previous example says: $\mathcal{L}^{-1}\left\{\dfrac{s+3}{(s+3)^2+4}\right\} = e^{-3t}\cos(2t)$

In searching for an inverse Laplace transform of this from, we must recognize that we have a shifted Laplace transform $F(s-a)$. We must identify $f(t)$ such that $\mathcal{L}\{f(t)\} = F(s)$. Then $\mathcal{L}^{-1}\{F(s-a)\}$ is the identified $f(t)$ times e^{at}.

In symbols: If $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}^{-1}\{F(s-a)\} = \mathcal{L}^{-1}\left\{F(s)\big|_{s\to s-a}\right\} = e^{at} f(t)$

Example:

a) $\mathcal{L}^{-1}\left\{\dfrac{7}{(s+2)^4}\right\}$

b) $\mathcal{L}^{-1}\left\{\dfrac{2s+5}{(s-3)^2}\right\}$

c) $\mathcal{L}^{-1}\left\{\dfrac{s+2}{s^2-4s-5}\right\}$

d) $\mathcal{L}^{-1}\left\{\dfrac{3s+5}{s^2+2s+7}\right\}$

Example: Use the Laplace transform to solve $y'' - 4y' + 4y = t^3 e^{2t}$, $y(0) = 0$, $y'(0) = 0$.