

§ 7.2 Inverse Transforms and Transforms of Derivatives

Introduction: In this section we will see how to use the Laplace transform to solve some simple DEs. It would be good to have your small table of transforms from 7.1 out.

If $F(s)$ represents the Laplace transform of a function $f(t)$, that is, $\mathcal{L}\{f(t)\} = F(s)$, then we say $f(t)$ is the **inverse Laplace transform** of $F(s)$ and we write $\mathcal{L}^{-1}\{F(s)\} = f(t)$. From section 7.1, we can write some inverse Laplace transforms.

a) $\mathcal{L}\{1\} = \frac{1}{s}$, then _____

b) $\mathcal{L}\{t^2\} = \frac{2}{s^3}$, then _____

c) $\mathcal{L}\{\cos 4t\} = \frac{s}{s^2 + 16}$, then _____

Note: Our table of transforms from 7.1 (and in the back of your text) also serves as a table of inverse transforms.

Example: Use the table to find the given inverse Laplace transform. $\mathcal{L}^{-1}\left\{\frac{2}{s^2 - 4}\right\}$

The inverse question: If $F(s) = \frac{-2s + 6}{s^2 + 4}$, who is $f(t)$ such that $\mathcal{L}\{f(t)\} = \frac{-2s + 6}{s^2 + 4}$?

Linearity of $\mathcal{L}^{-1}\{F(s)\}$: The inverse Laplace transform is also a linear transform. For constants α and β :

$$\mathcal{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha \mathcal{L}^{-1}\{F(s)\} + \beta \mathcal{L}^{-1}\{G(s)\}$$

where F and G are the transforms of some functions f and g .

Example: Find the inverse Laplace transform.

a) $\mathcal{L}^{-1}\left\{\frac{8}{s^3}\right\}$

$$\text{b) } \mathcal{L}^{-1} \left\{ \frac{1}{5s-2} \right\}$$

$$\text{c) } \mathcal{L}^{-1} \left\{ \frac{9s}{s^2-11} \right\}$$

$$\text{d) } \mathcal{L}^{-1} \left\{ \frac{-3s+7}{s^2+16} \right\}$$

Partial Fraction Decomposition:

State the form of the partial fraction decomposition for each rational expression:

$$\text{a) } \frac{3s+5}{(s-3)(s+4)}$$

$$\text{b) } \frac{2s^2+2s-1}{(s-1)(s+5)^3}$$

$$\text{c) } \frac{3s^2-5}{(s-2)(s^2+9)}$$

Example: Find the inverse Laplace transform.

$$\mathcal{L}^{-1} \left\{ \frac{s^2+5s+3}{(s-2)(s+4)(s+1)} \right\}$$

Transforms of Derivatives: It's now time to get back to differential equations. We've spent some time learning how to take Laplace transforms and how to take inverse Laplace transforms. Before proceeding into differential equations we will need to know how to take the Laplace transform of a derivative.

Assume f' is continuous on $[0, \infty)$.

Find $\mathcal{L}\{f'(t)\}$:

Find $\mathcal{L}\{f''(t)\}$:

Similarly, it can be shown that $\mathcal{L}\{f'''(t)\} = s^3 F(s) - s^2 f(0) - sf'(0) - f''(0)$, where $F(s) = \mathcal{L}\{f(t)\}$

We could generalize this process and write a nice concise theorem, but we really just need the above.

“ $\mathcal{L}\left\{\frac{d^n y}{dt^n}\right\}$ depends on $\mathcal{L}\{y(t)\}$ and the derivatives of $y(t)$ evaluated at $t = 0$.”

Solving Linear IVPs with Constant Coefficients:

Example: Use the Laplace transform to solve $\frac{dy}{dt} - y = 2 \cos(5t)$, $y(0) = 0$.

Example: Use the Laplace transform to solve $y'' - 4y' = 6e^{3t} - 3e^{-t}$, $y(0) = 1$, $y'(0) = -1$.