§ 7.1 Definition of the Laplace Transform

Introduction: In this chapter we will be looking at how to use Laplace transforms to solve differential equations. As we will see in a later section, we can use Laplace transforms to reduce a differential equation to an algebra problem.

Integral Transforms: Among the tools that are very useful for solving linear differential equations are integral transforms.

- Given a function $f(t)$, an integral transform is a relation of the form $\int_a^b K(s,t) f(t)\,dt$. This will be a function of __________ .
- We say the transform is taking us from the *t* domain to the *s* domain.
- The function $K(s,t)$ is called the **kernel** of the transform. We will be interested in a transform defined by an improper integral, $\int_0^\infty K(s,t) f(t)\,dt = \lim_{b\to\infty}\int_0^b K(s,t) f(t)\,dt$.
  If the limit **exists**, we say the integral exists or is **convergent**. If the limit **does not exist**, we say the integral does not exist or is **divergent**.
- Often times, the existence of the integral will depend on the value of *s*.

There are several integral transforms that are useful in applied mathematics, but in this chapter we consider only the Laplace Transform.

Definition 7.1.1: Laplace Transform
Let *f* be a function defined for $t \geq 0$. Then the integral

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

is said to be the **Laplace transform** of *f*, provided the integral converges.

Notation: If the integral exists, it will be a function of ________ . We write

$$\mathcal{L}\{f(t)\} = F(s) \qquad \mathcal{L}\{g(t)\} = G(s) \qquad \mathcal{L}\{y(t)\} = Y(s)$$

We will begin building a table of Laplace transforms. We'll start off with probably the simplest Laplace transform to compute.

Example: Evaluate $\mathcal{L}\{1\}$.

Example: Evaluate $\mathcal{L}\{t\}$.

Example: Evaluate $\mathcal{L}\{e^{-2t}\}$ and $\mathcal{L}\{e^{6t}\}$.

In general, $\mathcal{L}\{e^{at}\} =$ _______________

Example: Evaluate $\mathcal{L}\{t^n\}$, $n \in \mathbb{Z}^+$.

Example: Evaluate $\mathcal{L}\left\{\sin\left(kt\right)\right\}$.

Linearity of the Laplace transform: Show that $\mathcal{L}$ is a linear transform:

As long as both transforms converge for $s > c$ (for some c).

Example: Evaluate.

a) $\mathcal{L}\left\{1+8t\right\}$

b) $\mathcal{L}\left\{5e^{-2t}-3\sin 4t\right\}$

Theorem 7.1.1: Transforms of Some Basic Functions

a) $\mathcal{L}\{1\} = \dfrac{1}{s}$

b) $\mathcal{L}\{t^n\} = \dfrac{n!}{s^{n+1}}$, $n = 1, 2, 3, \ldots$

c) $\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}$

d) $\mathcal{L}\{\sin(kt)\} = \dfrac{k}{s^2+k^2}$

e) $\mathcal{L}\{\cos(kt)\} = \dfrac{s}{s^2+k^2}$

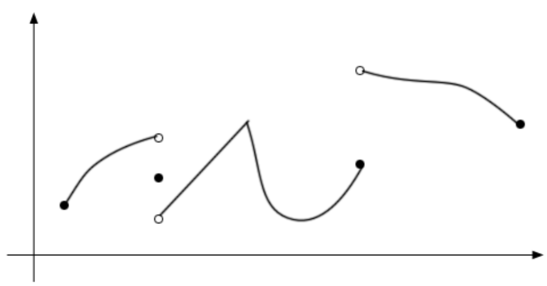
f) $\mathcal{L}\{\sinh(kt)\} = \dfrac{k}{s^2-k^2}$

g) $\mathcal{L}\{\cosh(kt)\} = \dfrac{s}{s^2-k^2}$

Sufficient Conditions for the Existence of $\mathcal{L}\{f(t)\}$:

When will this exist? What observations can we make about the function $f(t)$ that will guarantee the existence of its Laplace transform?

Definition: A function is called **piecewise continuous** on an interval if the interval can be broken into a finite number of subintervals on which the function is continuous on each open subinterval (*i.e.* the subinterval without its endpoints) and has a finite limit at the endpoints of each subinterval. Below is a sketch of a piecewise continuous function.



Definition 7.1.2: Exponential Order
A function *f* is said to be of **exponential order** if there exist constants *c*, $M > 0$, and $T > 0$ such that $|f(t)| \le Me^{ct}$ for all $t > T$.

The following functions are of exponential order: $y = 5$, $y = \cos(5t)$, $y = e^{5t}$

The function $y = e^{t^2}$ is not of exponential order.

Example: Show that $f(t) = t^5$ is of exponential order.

We must answer the question: Does there exist c, $M > 0$, and $T > 0$ such that $|f(t)| \le Me^{ct}$ for all $t > T$?

Theorem 7.1.2: Sufficient Conditions for Existence

If f is a piecewise continuous function on $[0,\infty)$ and of exponential order, then $\mathcal{L}\{f(t)\}$ exists for $s > c$.

For the proof, see page 228 in the text.

Example: Evaluate $\mathcal{L}\{f(t)\}$ where $f(t) = \begin{cases} 2t+1 & 0 \le t < 1 \\ 1 & t \ge 1 \end{cases}$

Theorem 7.1.3: Behavior of $F(s)$ as $s \to \infty$

If f is piecewise continuous on $[0,\infty)$ and of exponential order and $F(s) = \mathcal{L}\{f(t)\}$, then $\lim_{s\to\infty} F(s) = 0$.

This says that if we start with a function that is piecewise continuous and of exponential order, then it has a transform, and as you let $s \to \infty$ in that transform, the transform will go to 0.

This also says that a function such as $F(s) = \dfrac{s}{s+4}$ or $F(s) = 2$ are not the Laplace transforms of some piecewise continuous function of exponential order. It does not say that they are not Laplace transforms.