§ 6.2 Solutions About Ordinary Points

For the homogeneous linear second-order DE: $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$ (1)

or in standard form: $y'' + P(x)y' + Q(x)y = 0$ (2)

we have the following definition and theorem:

Definition 6.2.1: Ordinary and Singular Points
A point $x = x_0$ is said to be an **ordinary point** of the differential equation (1) if both coefficients $P(x)$ and $Q(x)$ in the standard form (2) are analytic at x_0. A point that is not an ordinary point of (1) is said to be a **singular point** of the DE.

Theorem 6.2.1: Existence of Power Series Solutions
If $x = x_0$ is an ordinary point of the DE (1), we can always find two linearly independent solutions in the form of a power series centered at x_0, that is, $y(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$
A power series solution converges at least on some interval defined by $|x - x_0| < R$, where R is the distance from x_0 to the closest singular point.

Notes:

- Sometimes you can find the summation notation for your solution, sometimes not (in which case you can just calculate the first so many coefficients). In the homework, feel free to simply write your solution in expanded form, do not worry about converting it into a sum, but be sure to show enough terms to show a pattern.
- We will, for the sake of simplicity, find only power series solutions about the ordinary point $x = 0$.

Strategy for Finding a Power Series Solution:

- Substitute $y = \sum_{n=0}^{\infty} c_n x^n$ into the DE
- Combine the series
- Equate all coefficients of x to 0 to determine c_n (Identity Property)
- This leads to two sets of coefficients so that we have two distinct power series $y_1(x)$ and $y_2(x)$
- The general solution of the DE is $y = c_0 y_1(x) + c_1 y_2(x)$

Example: Find two power series solutions of the given DE about the ordinary point $x = 0$. Compare your solution with how we already know how to solve the DE.

$$y'' + 2y' = 0$$

Example: Solve. $y'' - xy' + 2y = 0$

Example: Solve. $(x+1)y'' - (2-x)y' + y = 0$ subject to $y(0) = 2$ and $y'(0) = -6$