§ 6.1 Review of Power Series

Power Series: A **power series** is a series of the form $\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots$
Such a series is called a **power series centered at *a*** or a **power series about *a***.

Important Facts:

- **Convergence**: A power series is convergent at a specific value of x if its sequence of partial sums $\{S_N(x)\}$ converges. That is, $\lim_{N\to\infty} S_N(x) = \lim_{N\to\infty} \sum_{n=0}^{N} c_n (x-a)^n$ exists.
- **Interval of Convergence**: The interval of convergence is the set of all real numbers x for which the series converges. The center of the interval of convergence is the center a of the series.
- **Radius of Convergence**: The radius R of the interval of convergence of a power series is called its radius of convergence. The power series will converge if $|x-a| < R$ and diverge if $|x-a| > R$. If the series converges only at its center a, then $R = 0$. If the series converges for all x, then $R = \infty$. The inequality $|x-a| < R$ is equivalent to $a - R < x < a + R$. A power series may or may not converge at the endpoints $a - R$ and $a + R$.
- **Absolute Convergence**: Within its interval of convergence, a power series converges absolutely. That is, if x is in the interval of convergence and is not an endpoint of the interval, then the series of absolute values $\sum_{n=0}^{\infty} \left| c_n (x-a)^n \right|$ converges.

Determining the radius of convergence for most power series is usually quite simple if we use the ratio test.

The Ratio Test: Let $\sum a_n$ be a series and suppose $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

i) If $L < 1$, the series is *absolutely convergent* (and therefore convergent).
ii) If $L > 1$ or if $L = \infty$, the series is *divergent*.
iii) If $L = 1$, the Ratio Test is inconclusive.

Note: The ratio test is always inconclusive at an endpoint $a \pm R$.

Example: Find the interval and radius of convergence for $\sum_{n=0}^{\infty} \frac{(-1)^n}{3^n (n+1)} x^n$.

- **A Power Series Defines a Function**: $f(x)=\sum_{n=0}^{\infty} c_n (x-a)^n$ whose domain is the interval of convergence of the series. If the radius of convergence is $R>0$ or $R=\infty$, then f is continuous, differentiable, and integrable on the intervals $(a-R, a+R)$ or $(-\infty, \infty)$. Also, $f'(x)$ and $\int f(x)\,dx$ can be found by term-by-term differentiation and integration.

  If $f(x)=\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$, then

  $f'(x)=$ $\qquad\qquad\qquad\qquad$ $f''(x)=$

- **Identity Property**: If $\sum_{n=0}^{\infty} c_n (x-a)^n = 0$ for all x in some open interval, then $c_n =$_____ for all n.
- **Analytic at a Point**: A function f is said to be analytic at a point a if it can be represented by a power series $f(x)=\sum_{n=0}^{\infty} c_n (x-a)^n$ with either a positive or an infinite radius of convergence.

Some important power series (see page 234 in your text):

$\sum_{n=0}^{\infty} x^n =$__________ on $(-1,1)$

$\sum_{n=0}^{\infty} \frac{1}{n!} x^n =$__________ on $(-\infty, \infty)$

$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} =$__________ on $(-\infty, \infty)$

$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} =$__________ on $(-\infty, \infty)$

Example: Find the Maclaurin series for the function $f(x)=x^2 e^{-3x}$.

Shifting the Summation Index: Combining two or more summations as a single summation often requires a shift in the index of summation.

Example: Rewrite the expression using a single power series whose general term involves x^k.

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+2}$$

This next example gives us a preview of the method that will be used in section 6.2 to find solutions of linear second-order DEs.

Example: Find a power series solution $y = \sum_{n=0}^{\infty} c_n x^n$ of the linear first-order DE.

$$y' = xy$$