§ 5.1 Linear Models: Initial-Value Problems – Spring/Mass Systems

Preliminaries:

Hooke's Law: Hang a spring from the ceiling. Attach to the base of that spring a mass *m*. The amount of stretch (elongation) of that spring will depend on the spring's "stiffness" or strength. Different masses will stretch the same spring different lengths.

Hooke's law states that the spring exerts a restoring force F_s in the opposite direction of elongation and proportional to the amount of elongation. That is, $F_s = ks$, where k is a constant of proportionality called the **spring constant** and s is the distance stretched.

Example: If a mass weighing 20 pounds stretches a spring 1/4 foot, then:

Note the units of *k*:

How far would a mass weighing 25 pounds stretch the same spring?

Convention:

- We denote position *s* to be equilibrium.
- Units: W = mg, where mass is measured in slugs, kilograms, or grams and g = 32 ft/s², 9.8 m/s², or 980 cm/s², respectively

Undamped Motion: Pull the spring down (or lift it up) from rest, release the spring (gently or with a boost). Assuming no outside forces act on the spring (air, friction, damping forces), we expect the spring to vibrate/osculate about equilibrium. We want to describe that motion.

Newton's Second Law: At equilibrium, the weight *W* is balanced by the restoring force *ks*. So at equilibrium:

If the mass is displaced by an amount *x* from its equilibrium position, the restoring force of the spring is: ______

We can equate Newton's Second Law with the resultant force of the restoring force and the weight:

Note:

- The negative sign indicates that the restoring force of the spring acts opposite to the direction of motion.
- Let x (or x(t)) denote the displacement of the spring (at s, x = 0), displacements below equilibrium are positive (i.e., if the spring is below equilibrium, x > 0).



DE of Undamped Motion:

We call this simple harmonic motion or free undamped motion.

Initial conditions make sense here.

 $x(0) = x_0 \rightarrow \text{ initial starting position / displacement}$ $x'(0) = x_1 \rightarrow \text{ initial velocity boost.}$

If $x_0 > 0$ and $x_1 > 0$, then we have:

If $x_0 = 0$ and $x_1 < 0$, then we have:

Equation of Motion: Undamped

Solve
$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$
 for $x(t)$

The **period** of motion is $T = \frac{2\pi}{\omega}$. This is the time it takes to complete one full cycle. That is, from a maximum to the next maximum, minimum to next minimum, or some position (directed) to the next "same directed position."

Note, when x(t) is at a minimum value for a given *t*, where is the spring?

The frequency of motion is $f = \frac{1}{T} =$ ______. This is the number of cycles completed in one unit of time.

For example, if $T = \frac{1}{3}$, what is the frequency?

Example: Free Undamped Motion

A mass weighing 16 lb stretches a spring $\frac{8}{9}$ ft. There is no damping and no external forces acting on the system. At t = 0 the mass is released 6 inches above the equilibrium position with a downward velocity of 1 ft/sec. Determine the equation of motion.

Alternate Forms of x(t): Notice the amplitude is not apparently obvious from the form found above.

Example: Rewrite the solution from the previous example in the form $x(t) = A\sin(\omega t + \phi)$.

Graphical Interpretation: One of the graphs below is a graph of the physical position of the spring, the other is a graph of $x(t) = \frac{\sqrt{10}}{6} \sin(6t - 1.249)$. Which is which? Note the period and amplitude.



Possible questions:

a) At what times is the mass passing through equilibrium?

b) At what times is the mass passing through equilibrium heading up?

c) At what times is the mass at its extreme displacement on either side of the equilibrium position?

d) What is the position (or instantaneous velocity or acceleration) of the mass at any given time?

e) At what times will the mass be 0.25 feet above equilibrium?

Side Notes:

Aging Springs: We expect the spring to get weak. That is, for k to decrease over time. This leads to

$$m\frac{d^2x}{dt^2} + ke^{-\alpha t}x = 0$$
 (can't be solved with the methods from chapter 4)

Airy's Differential Equation: In the cold, it gets stiff faster (assume *k* increases linearly with time).

$$m\frac{d^2x}{dt^2} + ktx = 0$$
 (can't be solved with the methods from chapter 4)

Free Damped Motion: Only in a perfect vacuum could we attain free undamped motion. In reality, we have damping forces (air, lateral spring on a cart has friction or damping from brakes, you could suspend the spring in water, honey, etc., or you consider a dashpot that stops the door from slamming, or a mountain bike/car shock...).

We assume that the damping force is proportional to (a constant multiple of) the instantaneous velocity. That is:

Then, Newton's second law provides:

 β is called the damping constant (or damping coefficient).

Rearranging terms and dividing the above equation by m, we get the differential equation for free damped motion.

$$m\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = 0 \qquad \text{or} \qquad \frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0$$

where $2\lambda = _$ and $\omega^2 = _$.

Solve the equation:

Case 1: $\lambda^2 - \omega^2 > 0$

What does it mean for $\lambda^2 > \omega^2$?

The motion of the spring is described by:

This motion is smooth and nonoscillating.

Case 2: $\lambda^2 - \omega^2 = 0$

The motion of the spring in this case is described by:

Case 3: $\lambda^2 - \omega^2 < 0$

The motion of the spring in this case is described by:

Here, we have oscillatory motion, but the mass approaches equilibrium as time increases.

Example: A 4-foot spring measures 8 feet long after a mass weighing 8 pounds is attached to it. The medium through which the mass moves offers a damping force numerically equal to $\sqrt{2}$ times the instantaneous velocity. Find the equation of motion if the mass is initially released from the equilibrium position with a downward velocity of 5 feet per second. Find the time at which the mass attains its extreme displacement from the equilibrium position. What is the position of the mass at this instant?

Example: After a mass weighing 10 pounds is attached to a 5-foot spring, the spring measures 7 feet. This mass is removed and replaced with another mass that weighs 8 pounds. The entire system is placed in a medium that offered a damping force that is numerically equal to the instantaneous velocity.

a) Find the motion of the mass if the mass is initially released from a point 0.5 feet below the equilibrium with a downward velocity of 1 foot per second.

b) Express the equation in the form which would be equivalent to that found in the example on page 4.

c) Find the time at which the mass is first passing through equilibrium on the way down.

d) Below is a graph of the solution (along with two other graphs, who are they?).



Driven Motion: Now assume there is an external force acting on the vibrating mass and spring (think earthquake or oscillating ceiling to which the spring is attached, for the horizontal spring, you have a motor pushing the wheels). Call this external force f(t). Newton's second law gives the differential equation of **driven** or **forced** motion:

Dividing by *m* gives:

$$\frac{d^2x}{dt^2} + 2\lambda \frac{dx}{dt} + \omega^2 x = F(t)$$

where $2\lambda =$ _____, $\omega^2 =$ _____ and F(t) =_____.

This equation is nonhomogeneous so you will solve by either using the method of undetermined coefficients or variation of parameters.

Example: A mass of 1 slug is attached to a spring whose constant is 5 lb/ft. Initially, the mass is released 1 foot below the equilibrium position with a downward velocity of 5 ft/s. The motion takes place in a medium that offers a damping force that is numerically two times the instantaneous velocity.

a) Find the equation of motion if the mass is driven by an external force equal to $f(t) = 5\cos(3t)$.

Below is the graph of x(t) from part a):



Below is the graph of x(t) and the steady state term (still part a)):



b) Find the equation of motion if there is no damping force, the mass is driven by an external force equal to $f(t) = 5\cos(3t)$, subject to x(0) = 0 and x'(0) = 0.

In the interest of time, I may just give you the solution. But we will set up the equation and discuss the method of solving.

Solution:
$$x(t) = x_c + x_p = \frac{5}{4}\cos(\sqrt{5}t) - \frac{5}{4}\cos(3t)$$

Please note, the period of the spring/mass system with no forcing function is $T = \frac{2\pi}{\sqrt{5}}$.

Below is the graph of x(t) from part b):



c) "Find" the equation of motion if there is no damping force and the mass is driven by an external force equal to $f(t) = 5\cos\left(\frac{11}{5}t\right)$, subject to x(0) = 0 and x'(0) = 0.

Solution:
$$x(t) = x_c + x_p = -\frac{125}{4}\cos(\sqrt{5}t) + \frac{125}{4}\cos(3t)$$

Note: $\frac{11}{5} = 2.2$ and $\sqrt{5} \approx 2.23607$

Significance: the frequency of the forcing function is close to the frequency of the free undamped vibrations.

Consider a few graphs of $x(t) = -\frac{125}{4}\cos(\sqrt{5}t) + \frac{125}{4}\cos(3t)$.

