§ 4.6 Variation of Parameters

Introduction: Our goal in this section is to solve linear nonhomogeneous DEs with constant coefficients, where the input function $g(x)$ is not of a form discussed in section 4.4 (linear combinations of sines, cosines, polynomials and $e^{\alpha x}$). Our problem in 4.4 was that we could not find y_p when $g(x)$ was something of the form $\frac{1}{x}$, $\csc x$, $\ln x$, etc.

The method we examine in this section, due to the astronomer and mathematician Joseph Louis Lagrange, is known as **variation of parameters**.

Linear First-Order DEs Revisited:

Recall from section 2.3 Linear First-Order DEs:

$$a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)\rightarrow\frac{dy}{dx}+P(x)y=f(x)$$

Multiplying by the magical function $\mu(x)=e^{\int P(x)dx}$ had a magical effect:

$$e^{\int P(x)dx}\frac{dy}{dx}+e^{\int P(x)dx}P(x)y=e^{\int P(x)dx}f(x)\qquad\rightarrow\qquad\frac{d}{dx}\left[e^{\int P(x)dx}y\right]=e^{\int P(x)dx}f(x)$$

Then:

$$e^{\int P(x)dx}y=\int e^{\int P(x)dx}f(x)dx+c\qquad\rightarrow\qquad y=ce^{-\int P(x)dx}+e^{-\int P(x)dx}\int e^{\int P(x)dx}f(x)dx$$

This has the general form that we learned about in 4.1, $y=y_c+y_p$, where

- $y_c=ce^{-\int P(x)dx}$ is a solution of $\frac{dy}{dx}+P(x)y=0$
- $y_p=e^{-\int P(x)dx}\int e^{\int P(x)dx}f(x)dx$ is a particular solution of $\frac{dy}{dx}+P(x)y=f(x)$

We now try to find y_p a different way, using the method known as **variation of parameters**. The procedure is similar to what we did in section 4.2 (reduction of order).

Suppose we know that y_1 is a solution of $\frac{dy}{dx}+P(x)y=0$. It can be verified that $y_1=e^{-\int P(x)dx}$ is a solution, and because the equation is first-order, the general solution is $y=c_1y_1$.

Variation of Parameters: Try to find a particular solution y_p of the form $y_p=u_1(x)y_1(x)$. We have replaced the constant parameter c_1 with a function $u_1(x)$. Recall, we are trying to find u_1 so that $y_p=u_1(x)y_1(x)$ is a solution. Let's see if we can pull it off.

Substitute $y_p = u_1 y_1$ back into $\dfrac{dy}{dx} + P(x)y = f(x)$:

Linear Second-Order DEs:

Consider the linear second-order equation: $a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$ (1)

In standard form, this is: $y'' + P(x)y' + Q(x)y = f(x)$ (2)

where P, Q, and f are continuous on some interval I.

As we saw in 4.3, we can find the general solution to the associated homogeneous equation of (2) when the coefficients are constant; call it $y_c = c_1 y_1 + c_2 y_2$.

We now ask a question similar to above, can we replace the constants c_1 and c_2 with functions $u_1(x)$ and $u_2(x)$ so that $y_p = u_1 y_1 + u_2 y_2$ is a particular solution of (2)? We need to find those u functions!

Let's take some derivatives and substitute:

$$y_p' = u_1 y_1' + u_1' y_1 + u_2 y_2' + u_2' y_2$$

$$y_p'' = u_1 y_1'' + u_1' y_1' + u_1'' y_1 + u_1' y_1' + u_2 y_2'' + u_2' y_2' + u_2'' y_2 + u_2' y_2'$$

Substituting into (2) gives:

$$y_p'' + P(x)y_p' + Q(x)y_p =$$

$$= u_1y_1'' + 2u_1'y_1' + u_1''y_1 + u_2y_2'' + 2u_2'y_2' + u_2''y_2 + P(x)\left[u_1y_1' + u_1'y_1 + u_2y_2' + u_2'y_2\right] + Q(x)\left[u_1y_1 + u_2y_2\right]$$

$$= u_1\left[y_1'' + Py_1' + Qy_1\right] + u_2\left[y_2'' + Py_2' + Qy_2\right] + P\left[u_1'y_1 + u_2'y_2\right] + u_1''y_1 + u_1'y_1' + u_2''y_2 + u_2'y_2' + u_1'y_1' + u_2'y_2'$$

Since we are trying to find two unknown functions, we need two equations:

By Cramer's Rule, the solution of the system is:

$$u_1' = \frac{W_1}{W} \quad \text{and} \quad u_2' = \frac{W_2}{W}$$

where $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$, $W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}$, $W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$

Then, $u_1' = -\dfrac{y_2 f(x)}{W}$ and $u_2' = \dfrac{y_1 f(x)}{W}$.

Note: W is the Wronskian, and we know that $W\left(y_1(x), y_2(x)\right) \neq 0$ for all x in I because ________________________________ .

Summary of the Method:

Example: Solve. $y'' - 2y' + y = \dfrac{e^x}{1+x^2}$

Example: Solve. $2y'' + 18y = 6\tan(3x)$