§ 4.4 Undetermined Coefficients – Superposition Approach

Introduction: Our goal in this section is to solve nonhomogeneous linear differential equations of the form $a_n y^{(n)} + a_{n-1} y^{(n-1)} + ... + a_1 y' + a_0 y = g(x)$ (1)
where a_i, i = 1, 2, ,…, n are constants.

Strategy:

- Solve the corresponding homogeneous equation $a_n y^{(n)} + a_{n-1} y^{(n-1)} + ... + a_1 y' + a_0 y = 0$ using techniques from section 4.3. This will give us the complementary function y_c.
- Find a particular solution, y_p, of (1).
- Then our general solution of (1) is:

Method of Undetermined Coefficients:

This method is limited to linear DEs such as (1) where

- the coefficients a_i, i = 0, 1, …, n are constants and
- $g(x)$ is a constant k, a polynomial function, an exponential $e^{\alpha x}$, a sine or cosine function $\sin \beta x$ or $\cos \beta x$, or finite sums and products of these functions.

Consecutive derivatives of $y = \sin \beta x$ and $y = \cos \beta x$ yield: __________________________

Consecutive derivatives of $y = e^{\alpha x}$ yield: __________________________

Consecutive derivatives of polynomials yield: __________________________

Derivatives of sum and/or product combinations of the above functions (constants, polynomials, exponentials $e^{\alpha x}$, sines, and cosines) are again sums and products of constants, polynomials, exponentials $e^{\alpha x}$, sines, and cosines.

So when $g(x)$ is of this type, we can make a very educated guess as to the form of y_p, do some calculus and algebra, and find y_p. In this section $g(x)$ will be of the form:

In general, $g(x)$ will be of the form:

- $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ (n is a nonnegative integer)
- $P(x)e^{\alpha x}$
- $P(x)e^{\alpha x}\cos(\beta x)$ or $P(x)e^{\alpha x}\sin(\beta x)$ (α and β are real numbers)
- Any linear combination of the above functions

Example: Solve. $y'' + y' - 6y = 12x^2 + 3x - 1$

Example: Solve. $y'' - 4y' - 12y = \cos(2x)$

We can use the superposition principle from 4.1 to solve (1) when $g(x)$ is more than one form.

Example: Solve. $y'' - 8y' + 20y = 5x + 2 + 3xe^{2x}$ Note: $m^2 - 8m + 20 = 0 \rightarrow m = 4 \pm 2i$

Sometimes we may need to modify our choice for y_p.

Example: Solve. $y'' - 2y' - 3y = 5e^{3x}$

Case 1: No function in the assumed y_p is a solution of the associated homogeneous differential equation. In other words, no function in the assumed y_p is duplicated by a function in y_c.

Example: For each $g(x)$, determine the form of the "trial particular solution" y_p. (We are assuming there are no "duplicates.")

a) 2 (or any constant): $y_p =$ ______________________________
b) $3x+7$: $y_p =$ ______________________________
c) $5x^3-7x+1$: $y_p =$ ______________________________
d) $\cos(8x)$: $y_p =$ ______________________________
e) e^{9x}: $y_p =$ ______________________________
f) $(3x-8)e^{5x}$: $y_p =$ ______________________________
g) $2x^3e^{-4x}$: $y_p =$ ______________________________
h) $e^{2x}\sin(5x)$: $y_p =$ ______________________________
i) $3x^2\sin(5x)$: $y_p =$ ______________________________
j) $7xe^{3x}\cos(8x)$: $y_p =$ ______________________________

Example: Determine the form of a particular solution of

a) $y''+2y'+y=\sin x+3\cos(2x)$

b) $y''+9y=(x^2-3)\sin x$

c) $y''+3y'-10y=7xe^{3x}-2x^2+\cos(3x)$

Case 2: A function in the assumed particular solution is also a solution of the associated homogeneous differential equation.

Example: Find a particular solution of $y'' - 8y' + 16y = e^{4x}$.

Case 2 generalized:

1. Suppose $g(x)$ is made up of *m* terms of the kind in the first example on page 6.
2. Assume $y_p = y_{p_1} + y_{p_2} + ... + y_{p_m}$ is the form for the particular solution, where the y_{p_i}, $i = 1, 2, \ldots, m$ are the trial particular solutions for the respective *m* terms.
3. Under the circumstances for case 2 (a function in the assumed particular solution is also a solution of the associated homogeneous differential equation), we have a general rule:

   If any y_{p_i} contains terms that duplicate terms in y_c, then that y_{p_i} must be multiplied by x^n, where *n* is the smallest positive integer that eliminates the duplication.

Example: Solve. $y'' + 2y' - 24y = 16 - (x+2)e^{4x}$

Example: Determine the form of a particular solution of

a) $y''' + y'' = e^x \sin x + 2x$

b) $y^{(4)} - y'' = 4x + 2xe^{-x}$