§ 4.3 Homogeneous Linear Equations with Constant Coefficients

Introduction: In this section we consider equations of the form $ay'' + by' + cy = 0$ (1)
where a, b and c are constants. Let's assume that all solutions of this equation are of the form $y = e^{mx}$. Then

$y' =$ and $y'' =$

Plug these into equation (1) and we get

Since e^{mx} is never zero, the only way the above equation can be satisfied is if

$$am^2 + bm + c = 0$$

This equation is called the **characteristic equation** (or **auxiliary equation**) of equation (1).

Let m_1 and m_2 be the roots of this quadratic equation. We consider three cases:

- m_1 and m_2 real and distinct ($b^2 - 4ac > 0$)
- m_1 and m_2 real and equal ($b^2 - 4ac = 0$)
- m_1 and m_2 conjugate complex numbers ($b^2 - 4ac < 0$)

Case I: Distinct Real Roots

If the characteristic equation has two unequal real roots m_1 and m_2, we get two solutions $y_1 =$ __________ and $y_2 =$ __________ . These functions are linearly independent on $(-\infty, \infty)$ and hence form a fundamental set. The general solution of (1) on this interval is $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$.

Case II: Repeated Real Roots

When $m_1 = m_2$ we get one solution $y_1 = e^{m_1 x}$, where $m_1 = -\dfrac{b}{2a}$ (because $b^2 - 4ac = 0$). Using the formula we derived in section 4.2, we get a second solution:

$$y_2 = e^{m_1 x} \int \frac{e^{2m_1 x}}{e^{2m_1 x}} dx = e^{m_1 x} \int dx = x e^{m_1 x}$$

The general solution is then $y = c_1 e^{m_1 x} + c_2 x e^{m_2 x}$.

Note: $ay'' + by' + cy = 0 \to y'' + \dfrac{b}{a} y' + \dfrac{c}{a} y = 0$ so $P(x) = \dfrac{b}{a}$. Since $m_1 = -\dfrac{b}{2a}$ then $2m_1 = -\dfrac{b}{a}$ and

$$-\int P(x)\,dx = -\int \frac{b}{a} dx = \int 2m_1\,dx = 2m_1 x$$

Case III: Conjugate Complex Roots

If m_1 and m_2 are complex, then we can write $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$. We get two solutions

$y_1 =$ __________ and $y_2 =$ __________ . So $y = C_1 e^{(\alpha+i\beta)x} + C_2 e^{(\alpha-i\beta)x}$.

Since we started with only real numbers in our differential equation we would like our solution to only involve real numbers. To do this we'll need Euler's Formula: $e^{i\theta} = \cos\theta + i\sin\theta$

A nice variant of Euler's Formula that we'll need is: $e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$

So $e^{i\beta x} =$ and $e^{-i\beta x} =$

Adding these two equations gives:

Subtracting these two equations gives:

Since $y = C_1 e^{(\alpha+i\beta)x} + C_2 e^{(\alpha-i\beta)x}$ is a solution of (1) for any choice of C_1 and C_2, we can choose $C_1 = C_2 = 1$ for the first equation and $C_1 = 1$, $C_2 = -1$ for the second equation:

$$y_1 = e^{(\alpha+i\beta)x} + e^{(\alpha-i\beta)x} \qquad \text{and} \qquad y_2 = e^{(\alpha+i\beta)x} - e^{(\alpha-i\beta)x}$$

But $y_1 = e^{\alpha x}\left(e^{i\beta x} + e^{-i\beta x}\right) =$

and $y_2 = e^{\alpha x}\left(e^{i\beta x} - e^{-i\beta x}\right) =$

The last two results show that $e^{\alpha x}\cos\beta x$ and $e^{\alpha x}\sin\beta x$ are *real* solutions of (1). These solutions form a fundamental set on $(-\infty, \infty)$.

The general solution is then $y = c_1 e^{\alpha x}\cos\beta x + c_2 e^{\alpha x}\sin\beta x = e^{\alpha x}\left(c_1\cos\beta x + c_2\sin\beta x\right)$.

Example: Solve the following differential equations.

a) $3y'' + 2y' - 8y = 0$

b) $y'' + 14y' + 49y = 0$

c) $y'' - 4y' + 9y = 0$

Example: Solve the IVP. $y'' + 16y = 0$; $y\left(\frac{\pi}{2}\right) = -10$, $y'\left(\frac{\pi}{2}\right) = 3$

Higher-Order Equations:

Example: Solve the following differential equations.

a) $\frac{d^4y}{dx^4} - 7\frac{d^2y}{dx^2} - 18y = 0$

b) $y^{(5)} - y^{(4)} + 4y''' + 28y'' + 35y' + 13y = 0$