§ 4.2 Reduction of Order

Introduction: In 4.1 we saw that the general solution of a homogeneous linear second-order differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \qquad (1)$$

is a linear combination $y = c_1 y_1 + c_2 y_2$, where y_1 and y_2 are solutions that form a linearly independent set. If we know a nontrivial solution y_1 of the DE then (1) can be reduced to a linear first-order DE by means of a substitution involving the known solution y_1.

Reduction of Order:

Suppose that y_1 denotes a nontrivial solution of (1) and that y_1 is defined on an interval *I*. We seek a second solution y_2 so that y_1 and y_2 form a linearly independent set on the interval *I*. Recall that if y_1 and y_2 are linearly independent, then their quotient $\frac{y_2}{y_1}$ is nonconstant on *I* – that is, $\frac{y_2(x)}{y_1(x)} = u(x)$ or $y_2(x) = u(x)y_1(x)$. The function $u(x)$ can be found by plugging $y_2(x) = u(x)y_1(x)$ into the DE. This method is called **reduction of order** because it reduces the equation to a linear first-order DE.

Example: The indicated function $y_1(x)$ is a solution of the given DE. Use reduction of order to find a second solution $y_2(x)$.

$2x^2 y'' + xy' - 3y = 0$; $y_1 = x^{-1}$

General Case: Suppose we divide by $a_2(x)$ to put (1) in the **standard form**

$$y'' + P(x)y' + Q(x)y = 0 \qquad (2)$$

where $P(x)$ and $Q(x)$ are continuous on some interval *I*. Suppose further that $y_1(x)$ is a known solution of (2) on *I* and that $y_1(x) \neq 0$ for every *x* in *I*. If we define $y_2(x) = u(x)y_1(x)$, then

$$y' =$$

$$y'' =$$

So $y'' + Py' + Qy =$

This implies

If we let $w = u'$ then we have

This last equation is both linear and separable. Separating variables and integrating, we get

Solve the last equation for *w*, back substitute $w = u'$, and integrate again:

Choose $c_1 = 1$ and $c_2 = 0$

Therefore, a second solution of equation (2) is

Example: The indicated function $y_1(x)$ is a solution of the given DE. Use the formula we just derived to find a second solution $y_2(x)$.

$x^2 y'' - 3xy' + 5y = 0$; $y_1 = x^2 \cos(\ln x)$