§ 4.1 Preliminary Theory – Linear Equations

Introduction: We turn now to the solution of ODEs of order two or higher. Our goal is to find *all* solutions to linear ODEs of higher order. That is equivalent to finding the **general solution**, which is a family of solutions defined on some interval *I* containing *all* solutions of an ODE. In chapter 2, we were guaranteed we've found all solutions when solving first-order *linear* ODEs, for we were gifted the associated existence and uniqueness theorem.

Two important questions will be:
a) Do I have a solution?
b) Does it represent all solutions?

Initial-Value and Boundary-Value Problems

An ***n*^th^-order Linear Initial-Value Problem** is:

Solve: $$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+...+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$

Subject to: $$y(x_0)=y_0,\ y'(x_0)=y_1,\ \dots,\ y^{(n-1)}(x_0)=y_{n-1}$$

Existence and Uniqueness:

Theorem 4.1.1: Existence of a Unique Solution
Let $a_n(x),a_{n-1}(x),...,a_1(x),a_0(x)$ and $g(x)$ be continuous on an interval *I* and let $a_n(x)\neq 0$ for every *x* in this interval. If $x=x_0$ is any point in this interval, then a solution $y(x)$ of the initial value problem exists on the interval and is unique.

Using Theorem 4.1.1:

a) Consider the IVP: $4\frac{d^3y}{dx^3}+5\frac{d^2y}{dx^2}-2\frac{dy}{dx}+6y=0$, $y(2)=0$, $y'(2)=0$, $y''(2)=0$
Do you see the solution? Is it unique?

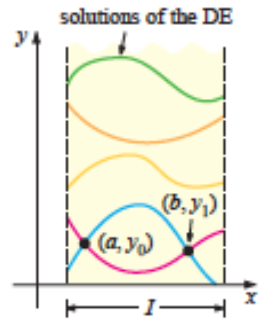
b) Consider the IVP: $y''-4y=12x$, $y(0)=4$, $y'(0)=1$

A solution is $y=3e^{2x}+e^{-2x}-3x$. Is this the only solution to the IVP on any interval containing $x = 0$?

c) Consider the IVP: $x^2 y'' - 2xy' + 2y = 6$, $y(0) = 3$, $y'(0) = 1$

A solution is $y = cx^2 + x + 3$. Is the solution unique?

Boundary-Value Problems: This type of problem will have a linear differential equation of order two or greater in which the function and/or derivatives are specified at *different* points, which we'll call boundary conditions.



Solve: $$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

Subject to: $y(a) = y_0$, $y(b) = y_1$

The values of $y(a) = y_0$ and $y(b) = y_1$ are called **boundary conditions**.

A solution of the BVP is a function satisfying the DE on some interval *I*, containing *a* and *b*, whose graph passes through the two points (a, y_0) and (b, y_1).

<u>**Homogeneous Equations**</u>

A linear *n*th-order differential equation of the form

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0 \qquad (1)$$

is said to be **homogeneous**. The equation

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \qquad (2)$$

(where $g(x)$ is not identically zero) is **nonhomogeneous**.

From now on we will make the following important assumptions about the above equations:
On some common interval *I*,

- The coefficient functions $a_i(x)$ and $g(x)$ are continuous
- $a_n(x) \neq 0$ for every x in the interval I

Differential Operators:

In calculus, differentiation is often denoted by the capital letter *D*: $\frac{dy}{dx} = Dy$. The symbol *D* is called a **differential operator**. For example, $D(\tan 3x) =$ ____________________ .

Higher-order derivatives can be expressed in terms of *D*:

$$\frac{d^2y}{dx^2} = D^2y \text{ and, in general } \frac{d^ny}{dx^n} = D^ny$$

In general, we define an ***n*th-order differential operator** to be

$$L = a_n(x)D^n + a_{n-1}(x)D^{n-1} + ... + a_1(x)D + a_0(x)$$

$$\text{so } L(y) \text{ means } L(y) = a_n(x)D^ny + a_{n-1}(x)D^{n-1}y + ... + a_1(x)Dy + a_0(x)y$$

We say that *L* is a **linear operator** because $L\{af(x) + bg(x)\} = aL(f(x)) + bL(g(x))$

Differential Equations:

Example: Consider the linear ODE $5x^2y'' - e^xy' + 4y = 2x - 8$. Write the DE in terms of the *D* notation.

We can rewrite equations (1) and (2) as

Superposition Principle: The next theorem tells us that the sum of two or more solutions of a homogeneous linear differential equation is also a solution.

Theorem 4.1.2: Superposition Principle – Homogeneous Equations
Let $y_1, y_2, ..., y_k$ be solutions of the homogeneous *n*th-order differential equation
$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + ... + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$ on an interval, *I*. Then the linear combination

$$y = c_1y_1(x) + c_2y_2(x) + ... + c_ky_k(x)$$

where the c_i, *i* = 1, 2, … , *k* are arbitrary constants, is also a solution on the interval.

Corollaries to Theorem 4.1.2

a) A constant multiple $y = c_1 y_1(x)$ of a solution $y_1(x)$ of a homogeneous linear differential equation is also a solution.
b) A homogeneous linear differential equation always possesses the trivial solution $y = 0$.

Using Theorem 4.1.2:

a) The functions $y_1 = x$ and $y_2 = x\ln x$ are both solutions of $x^2 y'' - xy' + y = 0$ on the interval $(0,\infty)$. By the superposition principle, ______________________________ is also a solution of the equation on the interval.

b) The function $y = e^{4x}$ is a solution of $y'' - 3y' - 4y = 0$. Some other solutions include ______________________________ .

Linear Dependence and Linear Independence:

Definition 4.1.1: Linear Dependence/Independence
A set of functions $f_1(x), f_2(x), ..., f_n(x)$ is said to be **linearly dependent** on an interval *I* if there exist constants $c_1, c_2, ..., c_n$, not all zero, such that

$$c_1 f_1(x) + c_2 f_2(x) + ... + c_n f_n(x) = 0$$

for every *x* in the interval. If the set of functions is not linearly dependent on the interval, it is said to be **linearly independent**.

In other words, a set of functions is linearly independent on an interval *I* if the only constants for which

$$c_1 f_1(x) + c_2 f_2(x) + ... + c_n f_n(x) = 0$$

for every *x* in the interval are $c_1 = c_2 = ... = c_n = 0$.

Let us first consider this idea for a set of two functions, $f_1(x)$ and $f_2(x)$. Assume the functions are linearly dependent on some interval:

We conclude that *a set of two functions* $f_1(x)$ *and* $f_2(x)$ *is linearly independent when neither function is a constant multiple of the other* on the interval.

Example: Is the set of functions $f_1(x) = 9\cos(2x)$ and $f_2(x) = 2\cos^2 x - 2\sin^2 x$ linearly dependent?

Example: The set of functions $f_1(x) = 1$, $f_2(x) = 2 + x$, $f_3(x) = 3 - x^2$, and $f_4(x) = 4x + x^2$ is *linearly dependent* on any interval *I* because …

Solutions of Differential Equations: We are interested in finding *linearly independent* solutions of a linear DE. It would be nice if we had a test for independence…

Definition 4.1.2: Wronskian
Suppose each of the functions $f_1(x), f_2(x), \ldots, f_n(x)$ possesses at least $n - 1$ derivatives. The determinant

$$W(f_1, f_2, \ldots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

where the primes denote derivatives, is called the **Wronskian** of the functions.

Theorem 4.1.3: Criterion for Linearly Independent Solutions
Let $y_1, y_2, \ldots, y_n$ be n solutions of the homogeneous linear nth-order differential equation (1) on an interval *I*. Then the set of solutions is **linearly independent** on *I* if and only if $W(y_1, y_2, \ldots, y_n) \neq 0$ for every x in the interval.

Definition 4.1.3: Fundamental Set of Solutions
Any set $y_1, y_2, \ldots, y_n$ of *n* linearly independent solutions of the homogeneous linear *n*th-order differential equation (1) on an interval *I* is said to be a **fundamental set of solutions** on the interval.

Theorem 4.1.4: Existence of a Fundamental Set
There exists a fundamental set of solutions for the homogeneous linear *n*th-order differential equation (1) on an interval.

Theorem 4.1.5: General Solution – Homogeneous Equations
Let $y_1, y_2, \ldots, y_n$ be a fundamental set of solutions of the homogeneous linear *n*th-order differential equation (1) on an interval *I*. Then the **general solution** of the equation on the interval is

where c_i, $i = 1, 2, \ldots, n$ are arbitrary constants.

Example: The functions $y_1 = e^{4x}$ and $y_2 = e^{-x}$ are both solutions of the homogeneous linear equation $y'' - 3y' - 4y = 0$ on the interval $(-\infty, \infty)$. Are the solutions linearly independent?

Thus, ___________________________________ is the general solution of the equation on the interval.

Example: The functions $y_1 = 1$, $y_2 = \cos t$ and $y_3 = \sin t$ are solutions of the homogeneous linear equation $y''' + y' = 0$ on the interval $(-\infty, \infty)$. Are the solutions linearly independent?

Thus, ___________________________________ is the general solution of the equation on the interval.

Nonhomogeneous Equations

Definition: Any function y_p, free or arbitrary parameters, that satisfies (2) is said to be a **particular solution** of the equation. For example, $y_p =$ _____ is a particular solution of the nonhomogeneous equation $y'' + 2y = 10$.

Now, let $y_1, y_2, ..., y_k$ be solutions of (1) and y_p a particular solution of (2), then is the linear combination $y = c_1y_1 + c_2y_2 + ... + c_ky_k + y_p$ a solution of (2)?

Well, recall the differential operator $L = a_n(x)D^n + a_{n-1}(x)D^{n-1} + ... + a_1(x)D + a_0(x)$.

$L(c_1y_1 + c_2y_2 + ... + c_ky_k) =$ and $L(y_p) =$

Then $L(c_1y_1 + c_2y_2 + ... + c_ky_k + y_p) =$

If we make *k* = *n*, then the linearly independent set of solutions $y_1, y_2, ..., y_n$ of the *n*th-order equation (1) can be used to build the general solution of (2).

Theorem 4.1.6: General Solution – Nonhomogeneous Equations
Let y_p be any particular solution of the nonhomogeneous linear *n*th-order differential equation (2) on an interval *I*, and let $y_1, y_2, ..., y_n$ be a fundamental set of solutions of the associated homogeneous differential equation (1) on *I*. Then the **general solution** of the equation on the interval is

$$y = c_1y_1(x) + c_2y_2(x) + ... + c_ny_n(x) + y_p$$

where c_i, *i* = 1, 2, …, *n* are arbitrary constants.

Proof: Let *L* be the differential operator $L = a_n(x)D^n + a_{n-1}(x)D^{n-1} + ... + a_1(x)D + a_0(x)$.

Let $Y(x)$ be a particular solution of (2). That is, ______________________________ .

Let $y_p(x)$ be a particular solution of (2). That is, ______________________________ .

Define a function $u(x) = Y(x) - y_p(x)$. Then, investigate $L(u)$:

Complementary Functions:

Notice, in order to solve the nonhomogeneous equation (2), we need to solve the associated homogeneous equation (1). If we call the solution of (1) $y_c = c_1 y_1(x) + c_2 y_2(x) + ... + c_n y_n(x)$ the **complementary function**, then the general solution of the nonhomogeneous differential equation (2) is

$$\begin{aligned} y &= \textit{complimentary function} + \textit{any particular solution} \\ &= c_1 y_1(x) + c_2 y_2(x) + ... + c_n y_n(x) + y_p \\ &= y_c + y_p \end{aligned}$$

Example: Consider the nonhomogeneous equation $2x^2 y'' + 5xy' + y = x^2 - x$. A particular solution is $y = \frac{1}{15}x^2 - \frac{1}{6}x$.

Consider the homogeneous equation $2x^2 y'' + 5xy' + y = 0$. The fundamental set of solutions on $(0, \infty)$ is $y_1 = x^{-1/2}$, $y_2 = x^{-1}$.

What is the **general solution** of $2x^2 y'' + 5xy' + y = x^2 - x$?

Another Superposition Principle:

Theorem 4.1.7: Superposition Principle – Nonhomogeneous Equations
Let $y_{p_1}, y_{p_2}, ..., y_{p_k}$ be *k* particular solutions of the nonhomogeneous linear *n*th-order differential equation (2) on an interval *I* corresponding, in turn, to *k* distinct functions $g_1, g_2, ..., g_k$. That is, suppose y_{p_i} denotes a particular solution of the corresponding differential equation

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + ... + a_1(x)y' + a_0(x)y = g_i(x)$$

where *i* = 1, 2, …, *k*. Then

$$y_p = y_{p_1}(x) + y_{p_2}(x) + ... + y_{p_k}(x)$$

is a particular solution of

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + ... + a_1(x)y' + a_0(x)y = g_1(x) + g_2(x) + ... + g_k(x)$$

Proof (for $n = 2$): Let L be the differential operator defined above and let $y_{p_1}(x)$ and $y_{p_2}(x)$ be particular solutions of the nonhomogeneous equations $L(y) = g_1(x)$ and $L(y) = g_2(x)$. If we let $y_p = y_{p_1}(x) + y_{p_2}(x)$, then $L(y_p) = L\{y_{p_1}(x) + y_{p_2}(x)\} = L(y_{p_1}(x)) + L(y_{p_2}(x)) = g_1(x) + g_2(x)$.

Example:

A particular solution of $y'' - 3y' + 4y = -16x^2 + 24x - 8$ is $y_{p_1} = -4x^2$

A particular solution of $y'' - 3y' + 4y = 2e^{2x}$ is $y_{p_2} = e^{2x}$

A particular solution of $y'' - 3y' + 4y = 2xe^x - e^x$ is $y_{p_3} = xe^x$

Find a solution of $y'' - 3y' + 4y = -16x^2 + 24x - 8 + 2e^{2x} + 2xe^x - e^x$.