§ 3.1 Linear Models

<u>Introduction</u>: We often want to describe the behavior of some real-life system or phenomenon, whether physical, sociological, or even economic, in mathematical terms. The mathematical description of a system of phenomenon is called a **mathematical model**.

Growth and Decay

In biological applications the rate of growth of certain populations (bacteria, small animals) is proportional to the population present at time t.

The IVP $\qquad \dfrac{dx}{dt} = kx$, $x(t_0) = x_0$

where k is some constant of proportionality, serves as a model for diverse phenomena involving either growth or decay. Problems describing growth (whether of populations, bacteria, or even capital) are characterized by a positive value of k, whereas problems involving decay (as in radioactive disintegration) have a negative value of k. So k is a either a **growth constant** ($k > 0$) or a **decay constant** ($k < 0$).

Example: The population of a community is known to increase at a rate proportional to the number of people present at time t. Suppose it is known that the initial population of the community has doubled in 5 years and that the population is 10,000 after 3 years. What was the initial population P_0? What will be the population in 10 years? How fast is the population growing at $t = 10$?

Radioactive Decay: To model radioactive decay, it is assumed that the rate $\dfrac{dA}{dt}$ at which the nuclei of a substance decay is proportional to the amount $A(t)$ of the substance remaining at time t: $\dfrac{dA}{dt} = kA$

Half-life: The half-life is a measure of the stability of a radioactive substance. The half-life is the time it takes for one-half of the atoms in an initial amount A_0 to disintegrate into the atoms of another element. Longer half-life implies greater stability.

Example: In 1986 the Chernobyl nuclear power plant exploded, and scattered radioactive material over Europe. Of particular note were the two radioactive elements iodine-131 and cesium-137. Suppose A_0 mg of iodine-131 was present initially. After 10 days, the mass had decreased by 58%. If the rate of decay is proportional to the amount of the substance present at time t,

a) find the half-life of iodine-131

b) find the amount remaining after 55 days if 300 mg were initially present

## Newton's Law of Cooling/Warming

Newton's law of cooling/warming states that the rate of change of the temperature of an object is proportional to the difference between its own temperature and the temperature of the surrounding medium (the ambient temperature). If we define

$T(t) =$ the temperature of the object at time t

$T_m =$ the temperature of the surrounding medium (ambient temperature)

then $\dfrac{dT}{dt} =$

Example: Suppose you cool a pot of soup in a 75^0 F room. Right when you take the soup off the stove, you measure its temperature to be 220^0 F. After 20 minutes, the soup has cooled to 170^0 F. Suppose you can eat the soup when it is 130^0 F. How long will it take to cool to this temperature?
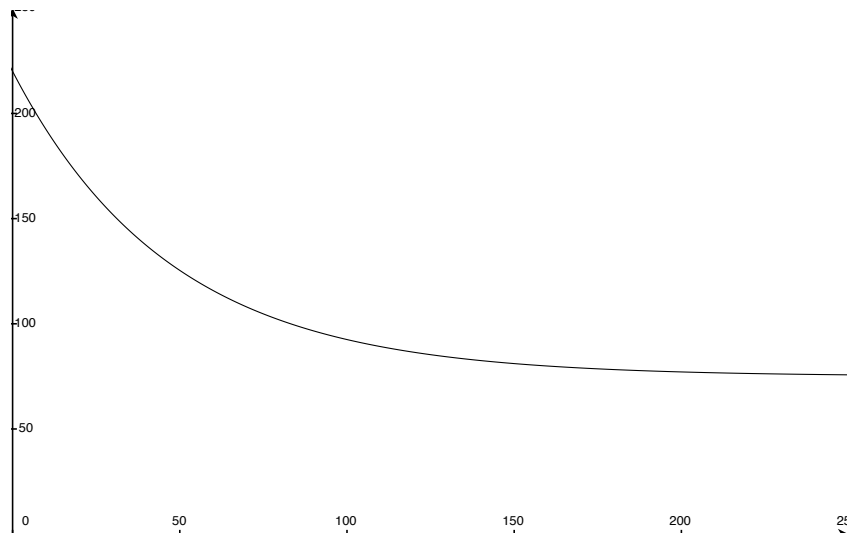
Mixtures

In these problems we will start with a substance that is dissolved in a liquid. Liquid will be entering and leaving a holding tank. The liquid entering the tank may or may not contain more of the substance dissolved in it. Liquid leaving the tank will of course contain the substance dissolved in it. If $A(t)$ gives the amount of the substance dissolved in the liquid in the tank at any time t we want to develop a differential equation that, when solved, will give us an expression for $A(t)$.

Suppose a giant mixing vat initially holds 300 gallons of brine (water in which a certain number of pounds of salt has been dissolved in it). Another brine solution is pumped into the tank at a rate of 3 gallons per minute; the concentration of the salt in this inflow is 2 pounds per gallon.

- We will assume the mixture is always well mixed, so the concentration of salt is uniform throughout the tank.
- We will also assume that the brine solution is leaving the tank at a rate of 3 gallons per minute (the same rate as the brine entering the tank).

If $A(t)$ denotes the amount of salt (measured in pounds) in the tank at time t, then the rate at which $A(t)$ changes is a net rate:

$$\frac{dA}{dt} =$$

R_{in} = (concentration of salt in liquid entering) × (input rate of brine) = input rate of salt

R_{out} = (concentration of salt in liquid exiting) × (output rate of brine) = output rate of salt

Concentration = $\dfrac{\text{Amount of salt in the tank at any time, } t}{\text{Volume of water in the tank at any time, } t}$

Example: Using the mixing tank from above, assume that there are initially 35 pounds of salt dissolved in the 300 gallons.

a) How many pounds of salt will be in the solution after 30 minutes? 3 hours?

b) How many pounds of salt will be in the solution after a long time?
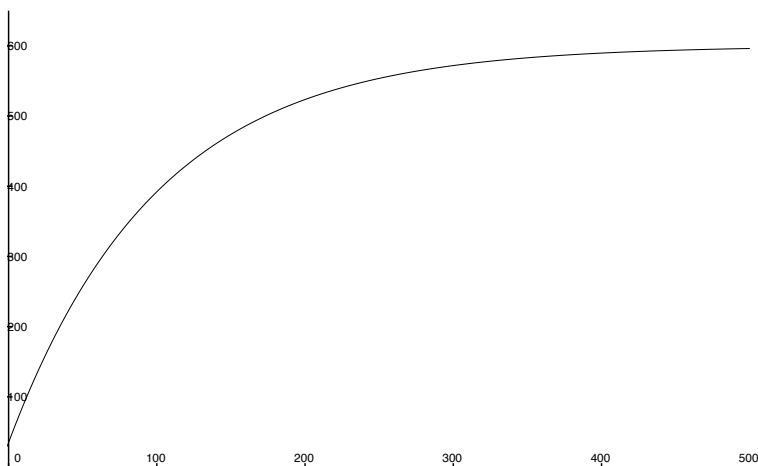
Example: Assume the same conditions as the previous example, but let the rate at which the solution is flowing out of the tank be 1 gallon per minute (this is slower than solution is being pumped in, so our tank is bigger than 300 gallons and is not yet full). Find the number of pounds of salt in the tank after 3 hours.