§ 2.4 Exact Equations

Recall from multivariable calculus, if $z = f(x, y)$ is a function of two variables with continuous partial derivatives, then its differential is

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

If $f(x, y) = c$, where c is a constant, then we have

Definition 2.4.1: Exact Equation
A differential expression $M(x, y)dx + N(x, y)dy$ is an **exact differential** in a region R of the xy-plane if it corresponds to the differential of some function $f(x, y)$ defined in R. A first-order differential equation of the form

$$M(x, y)dx + N(x, y)dy = 0$$

is said to be an **exact equation** if the expression on the left-hand side is an exact differential.

For example, $(2x + y^2)dx + (2xy)dy = 0$ is an exact equation because

The family of solutions would be ___________________ .

How do we know that an equation is exact?

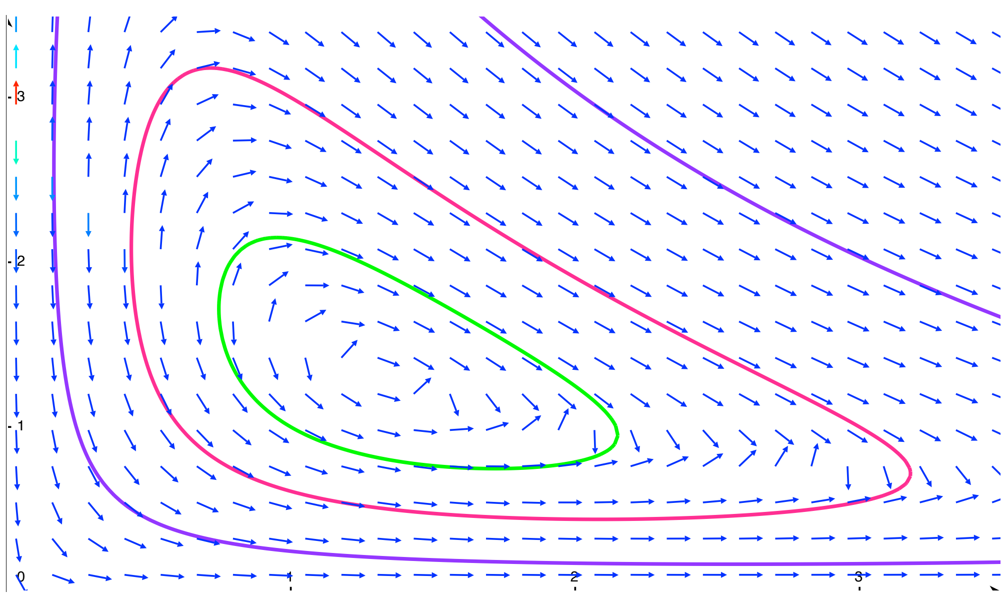
Notice in the above example, $\frac{\partial M}{\partial y} = 2y = \frac{\partial N}{\partial x}$. This is no coincidence.

Theorem 2.4.1: Let $M(x, y)$ and $N(x, y)$ be continuous and have continuous first partial derivatives in a rectangular region R: $a < x < b, c < y < d$. Then $M(x, y)dx + N(x, y)dy = 0$ is an exact differential equation in R if and only if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ at each point of R. That is, there exists a function f satisfying $\frac{\partial f}{\partial x} = M(x, y)$ and $\frac{\partial f}{\partial y} = N(x, y)$ if and only if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Example: Solve. $\left(3x^2y + xy^2\right)dx + \left(x^3 + x^2y\right)dy = 0$

Example: Solve. $y\cos x + 2xe^y + \left(\sin x + x^2e^y - 1\right)\dfrac{dy}{dx} = 0$

Example: Solve the IVP. $\left(1-\dfrac{3}{y}+x\right)\dfrac{dy}{dx}+y=\dfrac{3}{x}-1, \qquad y(3)=1$



Integrating Factors: Recall from 2.3 that the left-hand side of the linear equation $y' + P(x)y = f(x)$ can be transformed into a derivative when we multiply by some magical function $\mu(x)$.

We use a similar strategy when $M(x,y)dx + N(x,y)dy = 0$ is not exact. That is, we may be able to find an **integrating factor** $\mu(x,y)$ to make the equation exact:

$$\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0 \text{ or } (\mu M)dx + (\mu N)dy = 0$$

To find μ, we go back to the criteria for exactness: $(\mu M)_y = (\mu N)_x$

The product rule gives us:

(*)

The problem here is finding $\mu(x,y)$ because that would require us to solve a partial differential equation. To simplify things, suppose μ is a function of one variable; let's say that μ depends only on *x*. Then $\mu_x = \frac{d\mu}{dx}$ and $\mu_y = 0$, so we can rewrite the above equation (*) as:

$$\frac{d\mu}{dx} = \qquad (**)$$

We have another problem if $\frac{M_y - N_x}{N}$ depends on both *x* and *y*. However, if $\frac{M_y - N_x}{N}$ is only a function of *x*, then (**) is a first order ODE (separable and linear) which we can solve for μ. At last, we get

$$\mu(x) =$$

Similarly, if μ is a function of *y*, then $\frac{d\mu}{dy} = \frac{N_x - M_y}{M}\mu$. If $\frac{N_x - M_y}{M}$ is only a function of *y*, then we can solve for μ to get

$$\mu(y) =$$

Summary of Integrating Factors: Given $M(x,y)\,dx + N(x,y)\,dy = 0$ is <u>not</u> exact.

1. If $\dfrac{M_y - N_x}{N}$ is only a function of *x*, then multiplying the equation by $\mu(x) = e^{\int \frac{M_y - N_x}{N}\,dx}$ will make it exact.

2. If $\dfrac{N_x - M_y}{M}$ is only a function of *y*, then multiplying the equation by $\mu(y) = e^{\int \frac{N_x - M_y}{M}\,dy}$ will make it exact.

Example: Solve the IVP. $\left(y^2 + xy^3\right)dx + \left(5y^2 - xy + y^3 \sin y\right)dy = 0$, $y(1) = 1$