§ 2.3 Linear Equations

We continue our "quest" for solutions of first-order differential equations by examining a particularly "friendly" family of differential equations – *linear differential equations*.

Definition 2.3.1: Linear Equation

A first-order differential equation of the form

$$a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$

is said to be a **linear equation** in the variable *y*.

Standard Form: Divide both sides of the above equation by the coefficient $a_1(x)$ to get the linear equation in **standard form**:

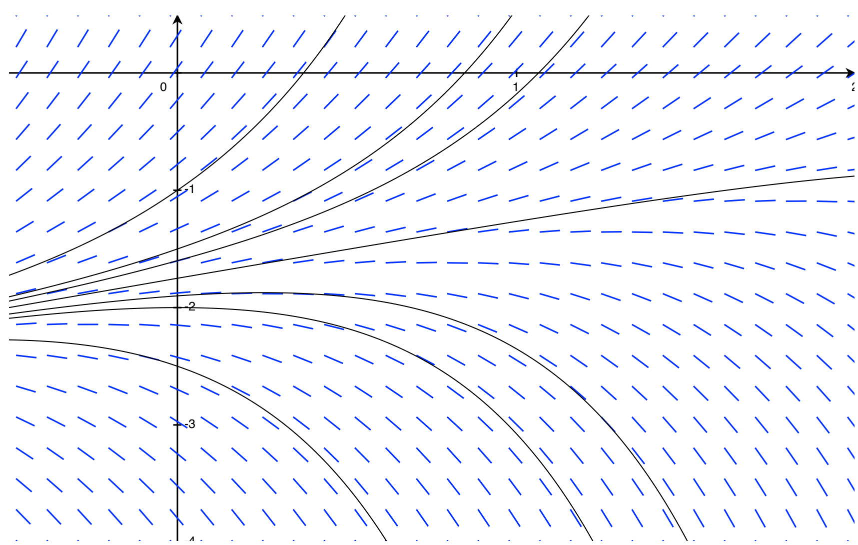
$$\frac{dy}{dx}+P(x)y=f(x)$$

Integrating Factors: To solve a linear DE we will use the fact that the left side can be transformed into the derivative of a product if we multiply the equation by a magical function $\mu(x)\ldots$

Solving a Linear First-Order Equation

1. Put the equation in *standard form*: $\dfrac{dy}{dx}+P(x)y=f(x)$
2. Identify $P(x)$ and find the *integrating factor* $\mu(x)=e^{\int P(x)dx}$. No constant of integration is needed when evaluating $\int P(x)dx$, i.e. let $c = 0$.
3. Multiply both sides of the equation by $\mu(x)$.
4. Verify the left side is the derivative of the product of $\mu(x)=e^{\int P(x)dx}$ and y and write it as such:
$$\underbrace{\frac{d}{dx}\left[e^{\int P(x)dx}y\right]}_{\frac{d}{dx}(\mu(x)y)}=e^{\int P(x)dx}f(x)$$
5. Integrate both sides of the equation and solve for y.

Example: Solve $y'-2y=4-x$. Give the largest interval I over which the general solution is defined.



Notes on Existence and Uniqueness:

1. Suppose the functions P and f are continuous on I. We have shown that $y = e^{-\int P(x)dx}\int e^{\int P(x)dx} f(x)\,dx + ce^{-\int P(x)dx}$ is a one-parameter family of solutions of $\frac{dy}{dx} + P(x)y = f(x)$ and *every solution of* $\frac{dy}{dx} + P(x)y = f(x)$ *defined on I is a member of this family*. We say $y = e^{-\int P(x)dx}\int e^{\int P(x)dx} f(x)\,dx + ce^{-\int P(x)dx}$ is the **general solution** of the DE on the interval I.

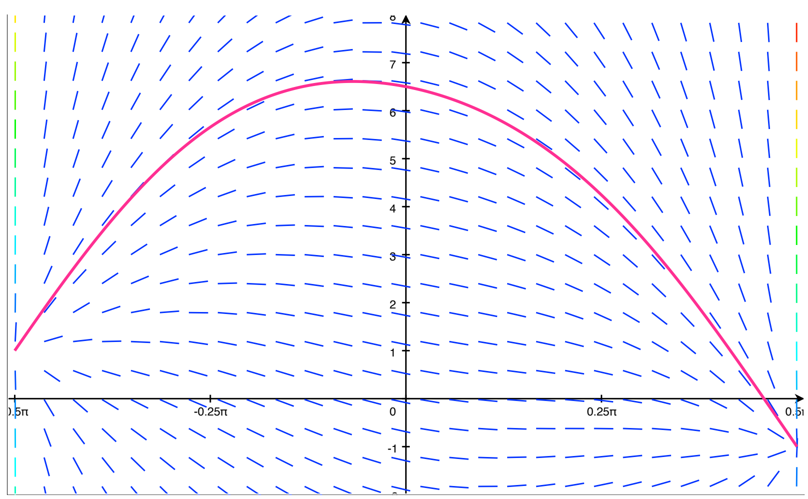
2. If we rewrite $\frac{dy}{dx} + P(x)y = f(x)$ in the normal form $y' = F(x,y)$, we have $F(x,y) = -P(x)y + f(x)$ and $\frac{\delta F}{\delta y} = -P(x)$. Since P and f are continuous on I, then F and $\frac{\delta F}{\delta y}$ are also continuous on I. We can conclude from Theorem 1.2.1 that the IVP

$$\frac{dy}{dx} + P(x)y = f(x); \quad y(x_0) = y_0$$

will have one unique solution, we just need to find the value of c in the general solution satisfying the initial condition.

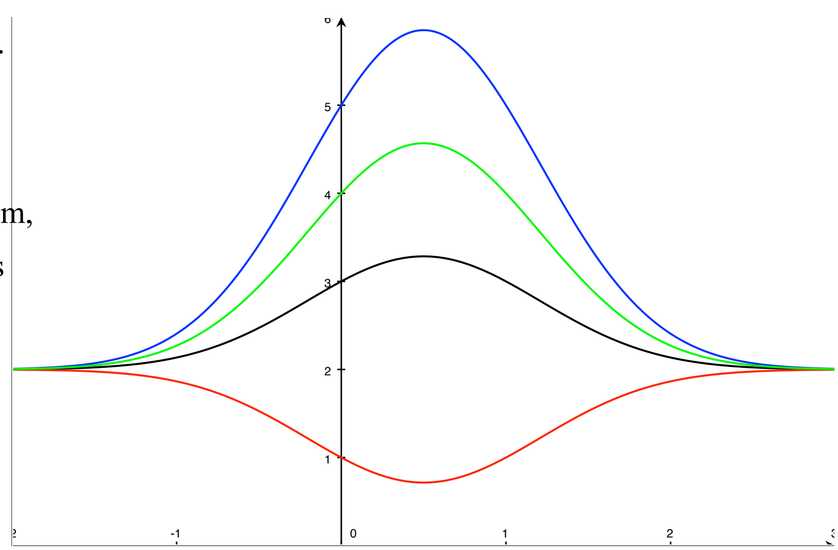
Example: Solve the IVP. Give the largest interval I over which the solution is defined.

$$(\cos x)\frac{dy}{dx} + (\sin x)y = 2\cos^3 x \sin x - 1: \quad y\left(\frac{\pi}{4}\right) = 3\sqrt{2}$$



Example: Find the general solution of $\frac{dP}{dt} + 2tP = P + 4t - 2$. Give the largest interval I over which the general solution is defined.

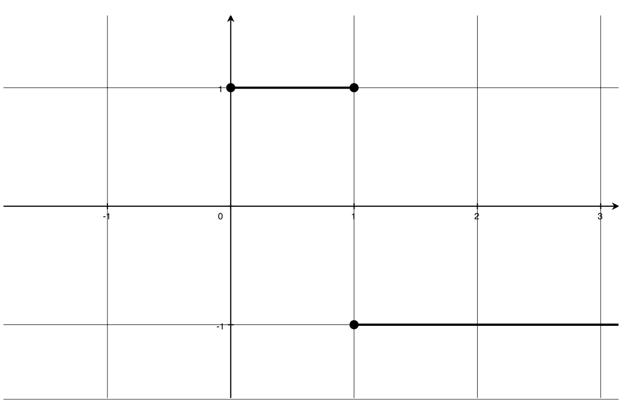
Solutions for $c = -1, 1, 2, 3$ are shown here. Note that as $t \to \infty$, $P \to$ ______ because $ce^{t-t^2} \to$_______. We call this term ce^{t-t^2} a **transient term**. Not all solutions have them, but they are worth noting in applications, as their contribution to the solution go to zero as the independent variable gets very large.



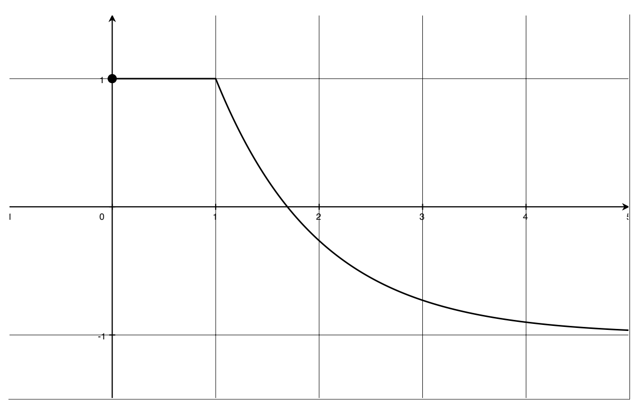
Example: Solve the IVP $\frac{dy}{dx}+y=f(x)$, $y(0)=1$, where $f(x)=\begin{cases} 1 & 0\le x\le 1 \\ -1 & x>1 \end{cases}$

Note: We want the solution to be continuous.

Graph of $f(x)$:



Graph of solution:



Example: Solve the IVP $ty' + 2y = 4t^2$, $y(1) = 2$