§ 2.1 Solution Curves Without a Solution

Introduction: The goal of this chapter is to present methods for solving various kinds of DE's. However, before taking up this task, we spend this section investigating a remarkable fact: It is possible to visualize and draw approximate graphs of the solutions of a DE without ever solving the equation. The tool that makes this visualization possible and allows us to explore the geometry of a DE is called the *direction field* (or *slope field*).

Direction Fields

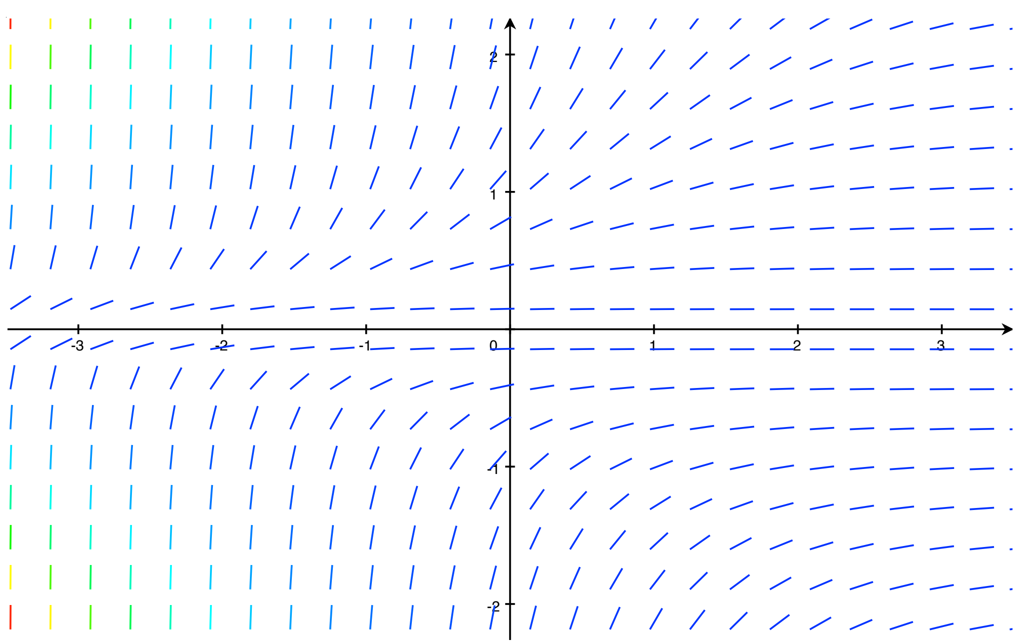
Consider the first-order ODE $\frac{dy}{dx} = f(x,y)$:

If we can find a solution curve $y = \phi(x)$ that satisfies the ODE, what does $f(x,y)$ tell us about the graph of the solution curve at (x,y)?

If a solution curve of this equation is displayed in the *xy*-plane, then the DE simply says that at each point (x,y) of the solution curve, the slope of the curve is $f(x,y)$. A **direction field** for the ODE can be constructed by evaluating *f* at each point of a rectangular grid of points in the *xy*-plane. At each point of the grid, a short line segment is drawn whose slope is the value of *f* at that point. Thus each line segment is tangent to the graph of the solution curve passing through that point. *A **direction field** is a picture that shows the slope of the solution curve at selected points of the xy-plane.*

Example: Below is the slope field for the DE $\frac{dy}{dx} = y^2 e^{-x}$. Sketch an approximate solution curve passing through each of the given points.

a) $y(2) = 1$ b) $y(-1) = 2$ c) $y(0) = -1$ d) $y\left(\frac{1}{2}\right) = 2$

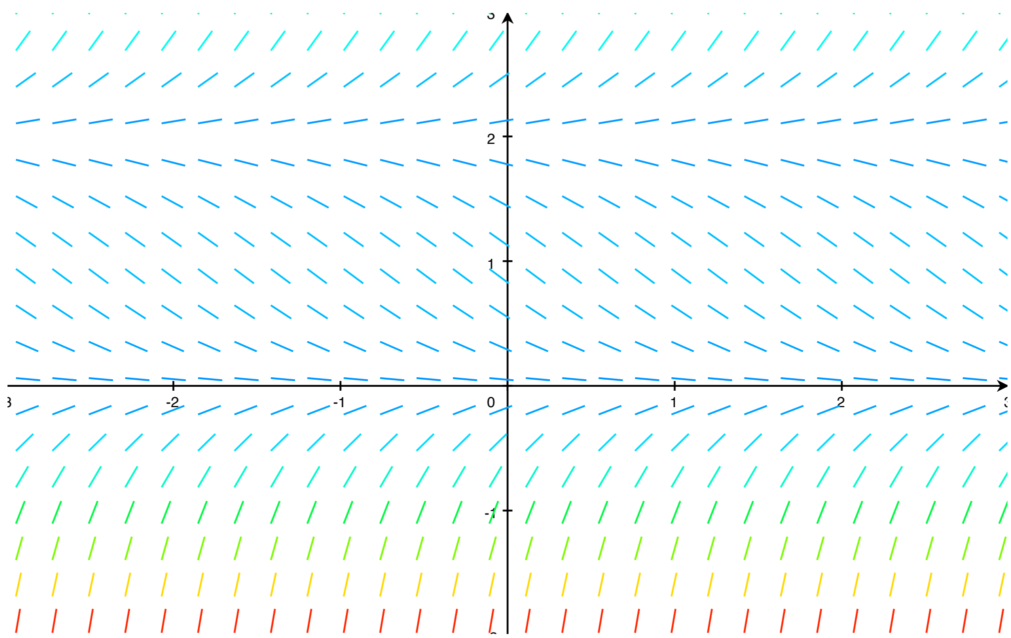


Example: Consider the IVP: $\dfrac{dy}{dx} = y(y-2),\ y(0)=1$

a) The general solution is $y = \dfrac{2}{1-ce^{2x}}$. Do you see a singular solution?

b) Find a solution of the IVP.

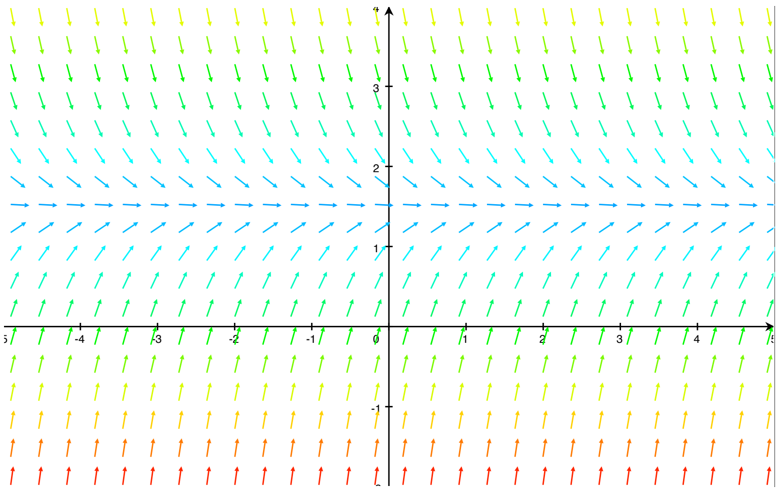
c) Sketch the solution curve ϕ to the IVP and the singular solution.



Note: The DE in the previous example is of the form $\dfrac{dy}{dx} = f(y)$, making it an **autonomous** first-order DE. The zeros of the function *f* are called **equilibrium points** (or **critical points**). If *c* is a critical point, then $y(x) = c$ is a constant (equilibrium) solution of the autonomous DE.

Example: A direction field is given for each differential equation. Based on the direction field, determine the behavior of y as $t \to \infty$. If this behavior depends on the initial value of y at $t = 0$, describe the dependency.

a) $\dfrac{dy}{dt} = 3 - 2y$



b) $\dfrac{dy}{dt} = (y+1)(y-2)$