§ 1.2 Initial-Value Problems

Introduction: We are often interested in problems in which we seek a solution $y(x)$ of a differential equation so that $y(x)$ satisfies certain conditions – that is, conditions on the unknown function $y(x)$ and its derivative at a point x_0.

Definition: An ***n*^th^-order Initial Value Problem (IVP)**

Solve: $\dfrac{d^n y}{dx^n} = f\left(x, y, y', \ldots, y^{(n-1)}\right)$

Subject to: $y(x_0) = y_0,\ y'(x_0) = y_1,\ \ldots,\ y^{(n-1)}(x_0) = y_{n-1}$

where $y_0, y_1, \ldots, y_{n-1}$ are arbitrary real constants. The values of $y(x_0) = y_0$, $y'(x_0) = y_1$, … , $y^{(n-1)}(x_0) = y_{n-1}$ are called the **initial conditions**.

Solving an n^{th}-order IVP frequently entails first finding an *n*-parameter family of solutions and then using the initial conditions at x_0 to determine the *n* constants in this family. The resulting particular solution is defined on some interval *I* containing x_0.

Geometric Interpretation of IVPs

First Order:

Solve: $\dfrac{dy}{dx} = f(x, y)$

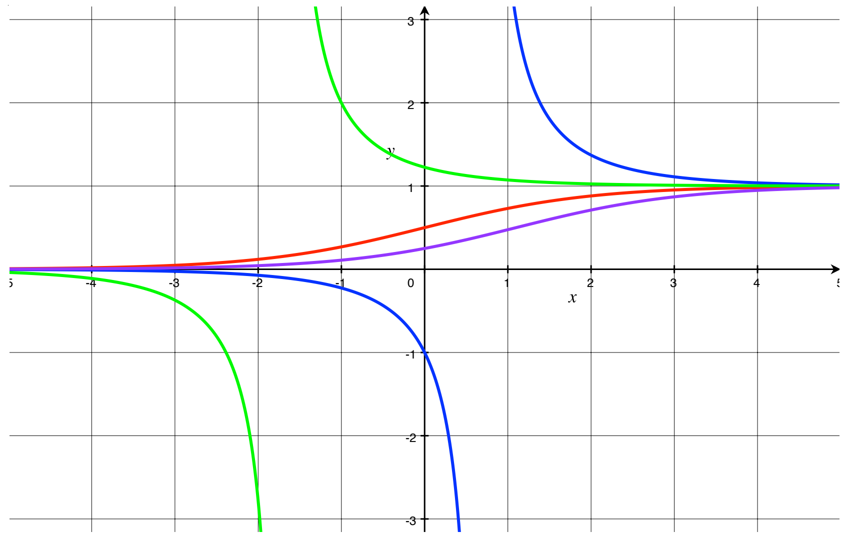
Subject to: $y(x_0) = y_0$

Second Order:

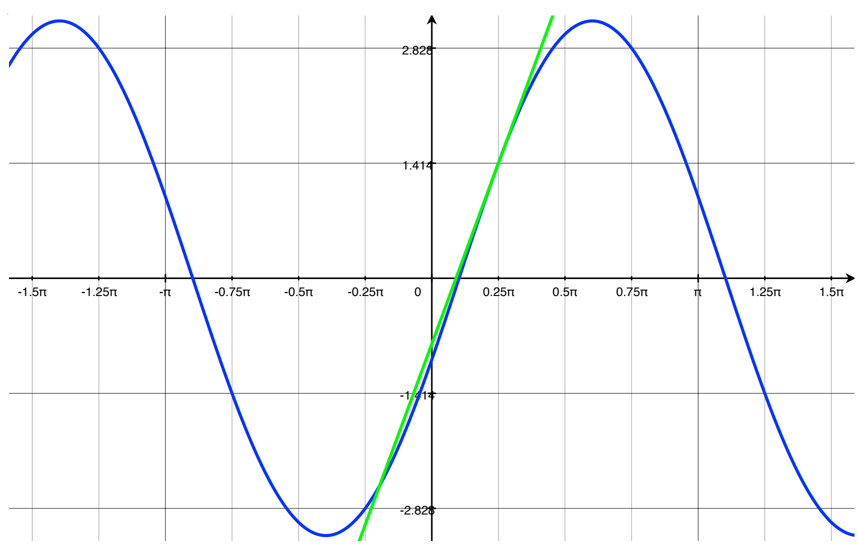
Solve: $\dfrac{d^2 y}{dx^2} = f(x, y, y')$

Subject to: $y(x_0) = y_0,\ y'(x_0) = y_1$

Example: $y = \dfrac{1}{1 + c_1 e^{-x}}$ is a one-parameter family of solutions of the first-order DE $y' = y - y^2$. Find a solution of the IVP subject to $y(-1) = 2$.



Example: $x = c_1 \cos t + c_2 \sin t$ is a two-parameter family of solutions of the second-order DE $x'' + x = 0$. Find a solution of the IVP subject to $x\left(\dfrac{\pi}{4}\right) = \sqrt{2}$, $x'\left(\dfrac{\pi}{4}\right) = 2\sqrt{2}$.

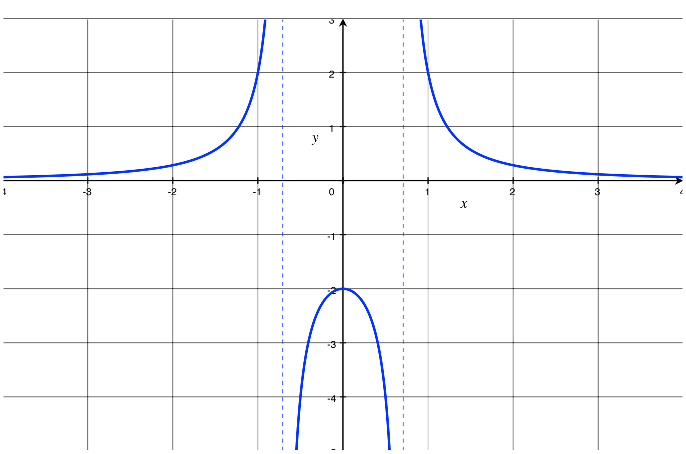


Example: $y = \dfrac{1}{x^2 + c}$ is a one-parameter family of solutions of the first-order DE $y' + 2xy^2 = 0$. Find a solution of the IVP subject to $y\left(\dfrac{1}{2}\right) = -4$.

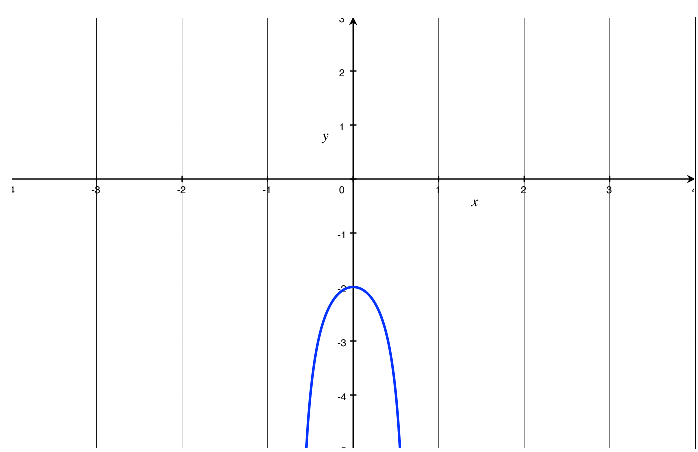
Solving gives $c =$ ________ . Thus, $y =$ ________________ .

- Considered as a *function*, the domain of $y = \dfrac{2}{2x^2 - 1}$ is ______________________ .
- Considered as a *solution of the DE* $y' + 2xy^2 = 0$, the interval I of definition could be any interval over which $y(x)$ is defined and differentiable. The largest intervals on which $y = \dfrac{2}{2x^2 - 1}$ is a solution are _________________________________ .
- Considered as a *solution of the IVP* $y' + 2xy^2 = 0$, $y\left(\dfrac{1}{2}\right) = -4$, the interval I of definition of $y = \dfrac{2}{2x^2 - 1}$ could be any interval over which $y(x)$ is defined, differentiable, *and* contains the initial point $y\left(\dfrac{1}{2}\right) = -4$. The largest interval for which this is true is ________________ .

Graph of $y = \dfrac{2}{2x^2 - 1}$



Solution curve of IVP



Existence and Uniqueness

The next two chapters focus on first-order ODEs. As we seek out to find solutions, some important questions arise.

1. Does a solution exist?
2. If so, is that solution unique?

Theorem 1.2.1: Existence of a Unique Solution
Consider the IVP:

Solve: $\dfrac{dy}{dx} = f(x,y)$ *Subject to*: $y(x_0) = y_0$

Let *R* be a rectangular region in the *xy*-plane defined by $a \le x \le b$, $c \le y \le d$ that contains the point (x_0, y_0) in its interior. If $f(x,y)$ and $\dfrac{\partial f}{\partial y}$ are continuous on *R*, then there exists some interval $I_0 : (x_0 - h, x_0 + h)$, $h > 0$, contained in $[a,b]$, and a unique function *y*(*x*), defined on I_0, that is a solution of the initial-value problem.

Note: The conditions in Theorem 1.2.1 are sufficient but not necessary. This means that when $f(x,y)$ and $\dfrac{\partial f}{\partial y}$ are continuous on a rectangular region *R*, then a solution exists and is unique whenever (x_0, y_0) is in *R*. However, if the conditions do not hold, then anything could happen.

Example: Determine a region of the *xy*-plane for which the DE $(1 + y^3)y' = x^2$ would have a unique solution whose graph passes through a point (x_0, y_0) in the region.