§ 1.1 Definitions and Terminology

Introduction: Many of the principles, or laws, underlying the behavior of the natural world are statements or relations involving rates at which things happen. When expressed in mathematical terms, the relations are equations and the rates are derivatives.

For example, a contagious disease (flu or virus) is spread throughout a community by people coming into contact with other people. Let $x(t)$ denote the number of people who have contracted the disease and $y(t)$ denote the number of people who have not been exposed. Consider the statement, "the rate at which the disease spreads is proportional to the number of *interactions* between these two groups of people." We can translate this statement into mathematical symbols:

This is an example of a *differential equation*.

Suppose you are given the equation $\frac{dy}{dx} = 2xy$ and asked, "what is the function y that satisfies this equation?" Answering this, and far more complex differential equations, is the focus of our course.

Definition 1.1.1: Differential Equation
An equation containing the derivatives of one or more unknown functions (or dependent variables), with respect to one or more independent variables, is said to be a ***differential equation*** **(DE).**

Classification by Type

Ordinary Differential Equation (ODE): This is a differential equation with only ordinary (as in the 1A/1B sense) derivatives of one or more unknown functions with respect to a single variable.

Partial Differential Equation (PDE): This is a differential equation involving partial derivatives of one or more unknown functions of two or more independent variables.

Examples of ODEs	Examples of PDEs

Notation

Leibniz notation: This has an advantage over the prime notation in that it clearly displays both the dependent and independent variables.

Prime notion: This notation is more compact but it loses the dependent variable. So we have to be careful and always keep track of who we are "taking derivatives with respect to."

Newton's dot notation: This is sometimes used to denote derivatives with respect to time *t*.

$$\frac{dy}{dt} = y' = \dot{y} \qquad \frac{d^2y}{dt^2} = y'' = \ddot{y} \qquad \frac{d^3y}{dt^3} = y''' = \dddot{y}$$

Classification by Order

The **order** of a differential equation is the order of the highest derivative in the equation.

For example:

$x\frac{d^3y}{dx^3} - \left(\frac{dy}{dx}\right)^4 + y = \sin x$ is a ________ - order ODE.

Normal Form: Consider the ODE $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} - 8y = e^{2t}$. This equation is equivalent to $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} - 8y - e^{2t} = 0$ or $y'' + 5y - 8y - e^{2t} = 0$. We can think of the LHS as a function of the 4 variables *t,* y, y' and y''. In general, we can write any n^{th}-order ODE in the form $F\left(x, y, y', \ldots, y^{(n)}\right) = 0$. (*F* is a function of ________ variables.) We assume now that we can solve for $y^{(n)}$ uniquely in terms of the remaining ________ variables. The differential equation becomes

(where *f* is a real-valued continuous function). We call this form the **normal form**.

Example: Write each equation in normal form. (Note the order of each equation.)

a) $xy' + 8y = x^5$

b) $y'' + 2y' - xy = 0$

First-order ODEs are occasionally written in **differential form** $M(x,y)dx + N(x,y)dy = 0$.

For example, consider $3x^2 y' + e^{2x} y = 5x$

Classification by Linearity

An n^{th} order ODE $F\left(x, y, y', ..., y^{(n)}\right) = 0$ is said to be ***linear*** if F is linear in $y, y', ..., y^{(n)}$.

For example:

Generally, we write the **linear *n*th-order ordinary differential equation** as:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + ... + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

or

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + ... + a_1(x)y' + a_0(x)y = g(x)$$

We recognize the characteristic properties of a linear ODE:

1. The function y and all its derivatives $y', y'', ..., y^{(n)}$ are all to the first power. There are no products of the function y.
2. The coefficients $a_n(x), a_{n-1}(x), ..., a_1(x), a_0(x)$ are functions of the one independent variable x.

First-order **linear** ODE (in general):

Second-order **linear** ODE (in general):

Example: Determine whether each equation is linear or non-linear. State its order.

a) $\cos t \dfrac{d^3 t}{du^3} + 4\left(\dfrac{dt}{du}\right)^7 + t = u^2$

b) $\cos u \dfrac{d^3 t}{du^3} + 4\dfrac{dt}{du} + t = u^2$

Solutions

Definition 1.1.2: Solution of an ODE
Any function ϕ, defined on an interval *I* and possessing at least *n* derivatives that are continuous on *I*, which when substituted into an n^{th}-order ODE reduces the equation to an identity, is said to be a ***solution*** of the equation on the interval.

Note:

1. Sometimes we will denote the solution $y(x)$
2. Given $y = e^{5x}$, then $\frac{dy}{dx} =$ __________ . Written another way, $\frac{dy}{dx} =$ __________ . Thus, the solution to the linear ODE $\frac{dy}{dx} = 5y$ is __________ .
3. The interval *I* referred to in 1.1.2 is sometimes called the ***interval of definition***, the ***interval of existence***, the ***interval of validity***, or the ***domain of the solution***. It can be of the form $(a,b), [a,b], (a,\infty)$, etc. (It is a portion of the real number line!) It is the part of the number line for which the solution function satisfies the DE. If you solve a DE, you must think about the interval of existence.

Example: Verify the indicated function is a solution of the given DE on the interval $(-\infty,\infty)$.

a) $\frac{dy}{dx} = xy^{1/2}$; $\quad y = \frac{1}{16}x^4$

b) $y'' - 10y' + 25y = 0$; $\quad y = xe^{5x}$

Note: Both equations in the previous example have the constant solution __________ . A solution of a DE that is identically 0 on an interval I is said to be a ***trivial solution***.

Solution Curve

If we graph the solution ϕ of an ODE, we call this graph a ***solution curve***. There may be a difference between the *graph of the function* ϕ and the *solution curve* ϕ. Because ϕ is a solution of a DE, it is differentiable and thus continuous. In other words, there may be a difference between the interval of definition for the solution to the ODE and the domain of the *function* ϕ.

Example: A solution to the ODE $\frac{dy}{dx} = 2xy^2$ is $y = \frac{1}{4 - x^2}$ (verify!). But y is not differentiable at $x = -2$ or $x = 2$, and is discontinuous at $x = -2$ or $x = 2$. A solution to a DE is a function that is defined *on an interval I* on which it is differentiable and satisfies the equation.

So a solution to $\frac{dy}{dx} = 2xy^2$ is $y = \frac{1}{4 - x^2}$ on any interval I that does not contain $x = -2$ or $x = 2$. We will take I to be as large as possible. Thus we take I to be either $(-\infty,-2),(-2,2)$ or $(2,\infty)$.

Explicit and Implicit Solutions

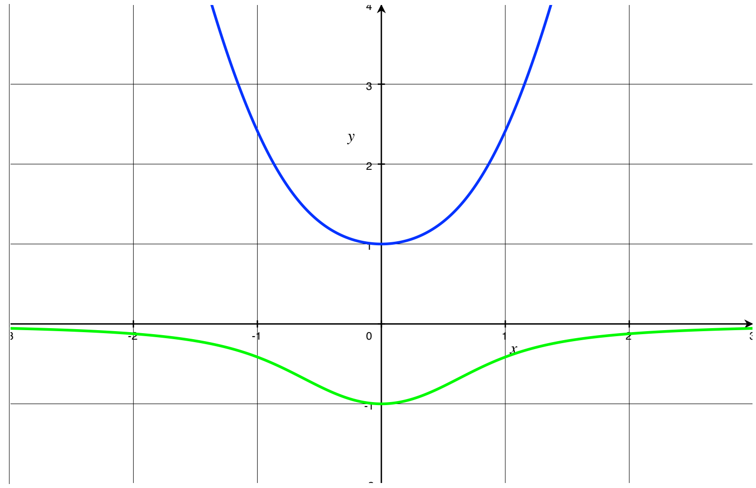
Recall from calculus implicitly and explicitly defined functions ($y = (x^2+1)\sqrt[3]{5x^2+2}$ versus $xe^y - 3y\sin x = 1$). A solution in which the dependent variable is expressed solely in terms of the independent variable and constants is said to be an **explicit solution**. All solutions from the previous two examples (including $y = 0$) were explicit solutions.

Definition 1.1.3: Implicit Solution of an ODE
A relation $G(x,y) = 0$ is said to be an ***implicit solution*** of an ordinary differential equation on an interval I, provided that there exists at least one *function* ϕ that satisfies the relation as well as the differential equation on I.

Example:

a) Verify that the relation $-2x^2y + y^2 = 1$ is an implicit solution of the ODE $2xy + (x^2 - y)\frac{dy}{dx} = 0$.

b) Find at least one explicit solution ϕ to the ODE
Give an interval of definition I of each solution ϕ.



Families of Solutions

When and why in integral calculus do we encounter a constant of integration, often called *c*.

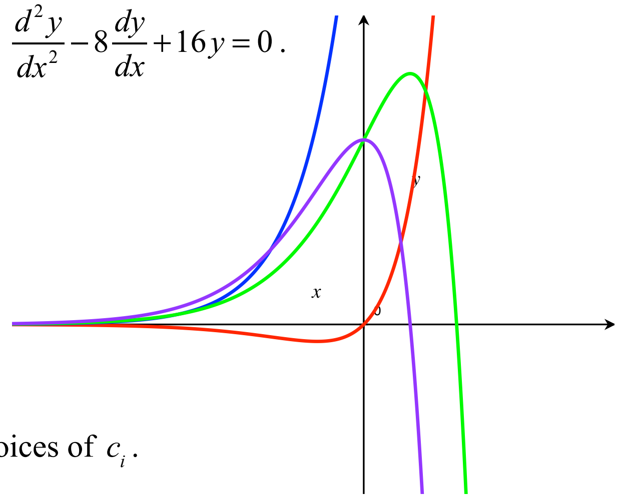
Recall, this gave us a family of curves.
Analogously, we will usually be encountering a constant or parameter *c* (or many of them) when solving ODEs.

1. When solving a *first-order* DE $F(x, y, y')=0$, we usually obtain a solution containing a single arbitrary constant or parameter *c*. If this is the case, the set of solutions $G(x, y, c)=0$ is called a **one-parameter family of solutions**.
2. When solving an n^{th}*-order* DE $F\left(x, y, y', \ldots, y^{(n)}\right)=0$, we seek an ***n*-parameter family of solutions** $G(x, y, c_1, c_2, \ldots, c_n)=0$. This means that a single DE can have an infinite number of solutions corresponding to different choices for the parameter(s) c_i.
3. A solution to a DE that has no parameters is called a **particular solution**.

Example: Consider the linear second-order differential equation $\dfrac{d^2y}{dx^2}-8\dfrac{dy}{dx}+16y=0$.

a) Circle the correct choice.
The (one/two)-parameter family $y=c_1e^{4x}+c_2xe^{4x}$ is an (explicit/implicit) solution to the ODE. (Verify!)

b) A ____________________ is $y=2e^{4x}-3xe^{4x}$ corresponding to $c_1=2$ and $c_2=-3$.



Graphed to the right are some particular solutions for various choices of c_i.

Singular Solution: Sometimes a DE has a solution that is not a member of a family of solutions of the equation. (In other words, no choice of c will produce this solution.)

For example, we have seen that $y = \frac{1}{16}x^4$ and $y = 0$ are solutions of $\frac{dy}{dx} = xy^{1/2}$. We will see in chapter two that the DE $\frac{dy}{dx} = xy^{1/2}$ has the one-parameter family of solutions $y = \left(\frac{1}{4}x^2 + c\right)^2$. If $c = 0$ we get $y = \frac{1}{16}x^4$. Notice that no choice of c will produce the trivial (and now singular) solution $y = 0$.

<u>Systems of Differential Equations</u>

A **system of ordinary differential equations** is two or more equations involving the derivatives of two or more unknown functions of a single independent variable. For example

$$\begin{aligned} \frac{dx}{dt} &= f(t,x,y) \\ \frac{dy}{dt} &= g(t,x,y) \end{aligned} \qquad\qquad \begin{aligned} \frac{dx}{dt} &= -5x - y + e^t \\ \frac{dy}{dt} &= 4x - 6y \end{aligned}$$

are examples of first-order systems of ODEs, where x and y denote dependent variables and t denotes the independent variable.

A **solution** of a system is a pair of differentiable functions $x = \phi_1(t)$ and $y = \phi_2(t)$, defined on a common interval I, that satisfy each equation of the system on this interval.