§ 2.3 Linear Equations

We continue our “quest” for solutions of first-order differential equations by examining a particularly “friendly” family of differential equations – *linear differential equations*.

**Definition 2.3.1: Linear Equation**

A first-order differential equation of the form

\[ a_1(x) \frac{dy}{dx} + a_0(x) y = g(x) \]

is said to be a **linear equation** in the variable \( y \).

**Standard Form:** Divide both sides of the above equation by the coefficient \( a_1(x) \) to get the linear equation in **standard form**:

\[ \frac{dy}{dx} + P(x) y = f(x) \]

**Integrating Factors:** To solve a linear DE we will use the fact that the left side can be transformed into the derivative of a product if we multiply the equation by a magical function \( \mu(x) \)…
Solving a Linear First-Order Equation

1. Put the equation in standard form: \( \frac{dy}{dx} + P(x)y = f(x) \)

2. Identify \( P(x) \) and find the integrating factor \( \mu(x) = e^{\int P(x) \, dx} \). No constant of integration is needed when evaluating \( \int P(x) \, dx \), i.e. let \( c = 0 \).

3. Multiply both sides of the equation by \( \mu(x) \).

4. Verify the left side is the derivative of the product of \( \mu(x) = e^{\int P(x) \, dx} \) and \( y \) and write it as such:

\[
\frac{d}{dx} \left[ e^{\int P(x) \, dx} y \right] = e^{\int P(x) \, dx} f(x)
\]

5. Integrate both sides of the equation and solve for \( y \).

**Example:** Solve \( y' - 2y = 4 - x \). Give the largest interval \( I \) over which the general solution is defined.
Notes on Existence and Uniqueness:

1. Suppose the functions $P$ and $f$ are continuous on $I$. We have shown that
   
   \[ y = e^{-\int P(x) \, dx} \int e^{\int P(x) \, dx} f(x) \, dx + ce^{-\int P(x) \, dx} \]

   is a one-parameter family of solutions of

   \[ \frac{dy}{dx} + P(x) y = f(x) \]

   and every solution of $\frac{dy}{dx} + P(x) y = f(x)$ defined on $I$ is a member of this family. We say $y = e^{-\int P(x) \, dx} \int e^{\int P(x) \, dx} f(x) \, dx + ce^{-\int P(x) \, dx}$ is the general solution of the DE on the interval $I$.

2. If we rewrite $\frac{dy}{dx} + P(x) y = f(x)$ in the normal form $y' = F(x, y)$, we have

   \[ F(x, y) = -P(x) y + f(x) \]

   and $\frac{\delta F}{\delta y} = -P(x)$. Since $P$ and $f$ are continuous on $I$, then $F$ and $\frac{\delta F}{\delta y}$ are also continuous on $I$. We can conclude from Theorem 1.2.1 that the IVP

   \[ \frac{dy}{dx} + P(x) y = f(x); \quad y(x_0) = y_0 \]

   will have one unique solution, we just need to find the value of $c$ in the general solution satisfying the initial condition.

**Example:** Solve the IVP. Give the largest interval $I$ over which the solution is defined.

\[ (\cos x) \frac{dy}{dx} + (\sin x) y = 2\cos^3 x \sin x - 1; \quad y \left( \frac{\pi}{4} \right) = 3\sqrt{2} \]
**Example:** Find the general solution of $\frac{dP}{dt} + 2tP = P + 4t - 2$. Give the largest interval $I$ over which the general solution is defined.

Solutions for $c = -1, 1, 2, 3$ are shown here. Note that as $t \to \infty$, $P \to \underline{\text{____}}$ because $ce^{-t^2} \to \underline{\text{____}}$. We call this term $ce^{-t^2}$ a **transient term**. Not all solutions have them, but they are worth noting in applications, as their contribution to the solution go to zero as the independent variable gets very large.
**Example:** Solve the IVP $\frac{dy}{dx} + y = f(x)$, $y(0) = 1$, where $f(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ -1 & x > 1 \end{cases}$

Note: We want the solution to be continuous.
Example: Solve the IVP \( ty' + 2y = 4t^2, \quad y(1) = 2 \)