CHAPTER 2
First-Order Differential Equations

§ 2.1 Solution Curves Without a Solution

Introduction: The goal of this chapter is to present methods for solving various kinds of DE’s. However, before taking up this task, we spend this section investigating a remarkable fact: It is possible to visualize and draw approximate graphs of the solutions of a DE without ever solving the equation. The tool that makes this visualization possible and allows us to explore the geometry of a DE is called the direction field (or slope field).

Direction Fields

Consider the first-order ODE \( \frac{dy}{dx} = f(x, y) \):

If we can find a solution curve \( y = \phi(x) \) that satisfies the ODE, what does \( f(x, y) \) tell us about the graph of the solution curve at \((x, y)\)?

If a solution curve of this equation is displayed in the \(xy\)-plane, then the DE simply says that at each point \((x, y)\) of the solution curve, the slope of the curve is \( f(x, y) \). A direction field for the ODE can be constructed by evaluating \( f \) at each point of a rectangular grid of points in the \(xy\)-plane. At each point of the grid, a short line segment is drawn whose slope is the value of \( f \) at that point. Thus each line segment is tangent to the graph of the solution curve passing through that point. A direction field is a picture that shows the slope of the solution curve at selected points of the \(xy\)-plane.

Example: Below is the slope field for the DE \( \frac{dy}{dx} = ye^{-x} \). Sketch an approximate solution curve passing through each of the given points.

a) \( y(2) = 1 \)  b) \( y(-1) = 2 \)  c) \( y(0) = -1 \)  d) \( y\left(\frac{1}{2}\right) = 2 \)
**Example:** Consider the IVP: \( \frac{dy}{dx} = y(y-2), \ y(0) = 1 \)

a) The general solution is \( y = \frac{2}{1-ce^{2x}} \). Do you see a singular solution?

b) Find a solution of the IVP.

c) Sketch the solution curve \( \phi \) to the IVP and the singular solution.

Note: The DE in the previous example is of the form \( \frac{dy}{dx} = f(y) \), making it an **autonomous** first-order DE. The zeros of the function \( f \) are called **equilibrium points** (or **critical points**). If \( c \) is a critical point, then \( y(x) = c \) is a constant (equilibrium) solution of the autonomous DE.
Example: A direction field is given for each differential equation. Based on the direction field, determine the behavior of \( y \) as \( t \to \infty \). If this behavior depends on the initial value of \( y \) at \( t = 0 \), describe the dependency.

a) \( \frac{dy}{dt} = 3 - 2y \)

b) \( \frac{dy}{dt} = (y + 1)(y - 2) \)