Chapter 6 - Rational Expressions and Equations

6.1 Introduction to Rational Functions and Expressions

REMARK 1. We will be making extensive use of the polynomial factoring that we learned in the previous section. If you are still having any difficult with that material please go back and look over it again.

Definition 1. A rational expression is an expression of the form \( \frac{p}{q} \), where \( p \) and \( q \) are polynomials and \( q \neq 0 \).

EX 1. The following are examples of rational expressions:

\[
\frac{3}{x}, \quad \frac{x + 2}{x}, \quad \frac{x^2 + 2x - 1}{x - 5}, \quad \frac{2xy - y}{x^2 - y^2}
\]

6.1.1 Domains of Rational Functions

Functions involving rational expressions are called rational functions. More specifically:

Definition 2. A rational function is a function of the form \( f(x) = \frac{p}{q} \) or \( y = \frac{p}{q} \), where \( p \) and \( q \) are polynomials and \( q \neq 0 \).

EX 2.

\[
f(x) = \frac{3}{x}, \quad y = \frac{3x^2 + 2x - 1}{x - 5}, \quad R(a) = \frac{a + 9}{a^2 - 4}
\]

With rational functions we need to be careful about the domain. Remember, the domain is the set of values that we can use to replace the independent variable.

Consider the first example from above \( f(x) = \frac{3}{x} \) can we let \( x = 0 \)? If not, why not?

In general, when dealing with a rational function, the domain is all the numbers except those where the denominator is zero.

REMARK 2. To find the domain of a rational function, find all values which make the denominator zero. The domain is then all numbers except those found.

EX 3. Find the domain of the following rational functions.

1. \( f(x) = \frac{x^2}{x^2 - 9} \)

2. \( g(y) = \frac{x - 4}{y^2 - x - 12} \)

REMARK 3. Note that we didn’t use the numerator at all when finding the domain.
6.1.2 Simplifying Rational Expressions

It will be important that we simplify our rational expressions completely.

**Definition 3.** A rational expression is simplified when the numerator and denominator have no common factors other than one.

**Remark 4.** This concept and process is identical to the concept of simplifying a fraction.

We have the following two step process to simplify a rational expression:

1. Factor both the numerator and denominator as completely as possible.
2. Divide both the numerator and denominator by any common factors.

**Ex 4.**

1. Simplify \( \frac{x^2 - 3x - 18}{x - 6} \)

2. Simplify \( \frac{4x^4 - 4x^3}{x^4 - x^2} \)

3. Simplify \( \frac{125x^3 - 8}{2 - 5x} \)

**Common Errors** Watch out for the following common errors:

6.1.3 Multiplying and Dividing Rational Expressions

To multiply rational expression we multiply just like we do with fractions, however to ensure our result is completely simplified we follow these steps:

1. Factor all the numerators and denominators completely.
2. Divide out the common factors.

We then multiply the numerators and denominators.
EX 5. -

1. Multiply: \( \frac{4x-5}{x-6} \cdot \frac{x^2-12x+36}{5-4x} \)

2. Multiply: \( \frac{x^2-2y}{x-y} \cdot \frac{2x^2-xy-y^2}{2x^2+3xy+y^2} \)

3. Multiply: \( \frac{mr+ms-nr-ns}{mr+ms+nr+ns} \cdot \frac{m^2+ms+mn+ns}{m^2-mn+ms-ns} \)

To divide rational expressions it is just like dividing fractions. We just multiply by the reciprocal.

EX 6. -

1. Divide: \( \frac{39x^5}{9y^7} \div \frac{13x^6}{4y^2} \)

2. Divide: \( \frac{6a^2-7a+2}{3a} \div \frac{3a^2-a-2}{2a^2+a} \)

3. Divide: \( \frac{m^4-n^4}{m^2+mn^2} \div \frac{m-n}{m^2-mn} \)
6.2 Addition and Subtraction of Rational Expressions

In the last section we learned how to multiply and divide rational expressions. We now shift to addition and multiplication. As with fractions, addition and subtraction is actually tougher than multiplication and division, this is because we will need to first find a common denominator.

6.2.1 With Common Denominators

If our expressions have a common denominator, we proceed similar to how we add or subtract fractions, we simply add or subtract the numerators and then simplify. **General Rule:** To add or subtract rational expressions:

\[
\frac{a}{c} + \frac{b}{c} = \frac{a + b}{c} \quad \frac{a}{c} - \frac{b}{c} = \frac{a - b}{c}
\]

Remember to simplify after adding or subtracting.

**EX 7.** -

1. Add: \(\frac{x^2 - 8x - 1}{(x+4)(x-6)} + \frac{3x-5}{(x+4)(x-6)}\)

2. Subtract: \(\frac{x}{(x+3)} - \frac{x^2 + 2x - 6}{x+3}\)

**Caution:** When subtracting be sure to subtract the entire numerator.

6.2.2 Finding the Least Common Denominator (LCD)

In order to add and subtract rational expressions with unlike denominators, it will be necessary to first find a common denominator. For rational expressions we will use the least common denominator (LCD).

**EX 8.** -

1. To motivate what follows let’s look at how we can find the LCD with fractions. Let’s add:

\[
\frac{1}{12} + \frac{1}{18}
\]
2. Find the LCD of \( \frac{1}{5x^2y} - \frac{8}{3x^3y^3} \)

3. Find the LCD of \( \frac{11}{x^2(x+1)} + \frac{13z}{x^2(x+1)^2} \)

4. Find the LCD of \( \frac{5}{4x^2-12x} - \frac{7}{x^2-6x+9} \)

5. Find the LCD of \( \frac{11x}{x^2+6x+5} + \frac{2x-3}{x^2+x-20} \)

### 6.2.3 Add and Subtract Expressions with Unlike Denominators

**Steps:**

1. Find the LCD
2. Rewrite each fraction as an equivalent fraction with the LCD
3. Leave denominator factored, multiply out the numerator.
4. Add or subtract
5. If possible, simplify the results.

**EX 9.**

1. Add: \( \frac{5}{m} + \frac{2}{n} \)
2. Add: $\frac{2}{3a^2} + \frac{13}{9ab^4}$

3. Subtract: $\frac{x^3}{x-6} - \frac{x+1}{x+6}$

4. Add: $\frac{3}{x-5} + \frac{x-5}{5-x}$

5. Subtract: $\frac{2x+1}{3x^2-2x-8} - \frac{3x-2}{2x^2-x-6}$

6. Simplify: $\frac{x+3}{x+1} + \frac{x-3}{x-1} - \frac{x+4}{x-1}$

### 6.3 Complex Fractions

A **complex fraction** is one that has a rational expression in its numerator or denominator or both.

**EX 10.** The following are all examples of complex fractions:

$$\frac{2}{5}, \quad \frac{x+2}{3x}, \quad \frac{7 + \frac{1}{z}}{1/z + \frac{3}{z}}$$

Our goal will be to simplify complex fractions, which means to rewrite it so it is not complex.

#### 6.3.1 Simplify Complex Fractions by Multiplying by a the LCD

One way to simplify these messes is to multiply the numerator and denominator by the LCD. Specifically:

**To Simplify:**
1. Find the LCD of all the fractions appearing within the complex fraction.
2. Multiply the numerator and denominator of the complex fraction by the LCD found in step 1.

**EX 11.**

1. Simplify \( \frac{x^2 - 2}{x^2} \)

2. Simplify \( \frac{a + \frac{7}{b}}{b + \frac{7}{a}} \)

**6.3.2 Simplify by Simplifying the Numerator and Denominator**

**EX 12.** Simplify: \( \frac{5x^2 - 2}{3x^2 - 4x} \)

**6.4 Solving Rational Equations**

A **rational equation** is an equation that contains at least one rational expression.

**Steps to solving a rational equation:**

1. Determine the LCD of all rational expressions.
2. Multiply both sides of the equation by the LCD.
3. Combine like terms on either side.
4. Solve using methods discussed before. (Note: if the previous steps were done properly, there should be no rational expressions left at this point).
5. Check the solutions found in the original equations.

**EX 13.** Solve: \( \frac{4x}{5} + \frac{3}{2} = \frac{x+1}{5} \)
REMARK 5. If a variable appears in the denominator of the original expression, it is important to check the solution. If the solution found makes the denominator equal to 0, it is not a solution. It is called an extraneous solution.

EX 14. -

1. Solve: \( \frac{20}{x^2+x-3} - \frac{3}{x+3} = \frac{4}{x-1} \)

2. Solve: \( x + \frac{7}{x} = 8 \)

3. Solve: \( \frac{4x}{x^2-9} + \frac{2}{x+3} = \frac{5}{x-3} \)