

## § 2.6 Euler's Method

Up to this point practically every differential equation that we've been presented with could be solved. The problem with this is that these are the exceptions rather than the rule. The vast majority of first order differential equations can't be solved analytically.

Therefore, it is important to be able to approach the problem in other ways. As we have seen, one way is to draw a direction field for the DE (which does not involve solving the equation) and then visualize the behavior of the solutions. The problem with this approach is that it's only really good for getting general trends in solutions and for long-term behavior of solutions. There are times when we will need something more. For instance, maybe we need to determine how a specific solution behaves, including some values that the solution will take. There is also a fairly large set of differential equations that are not easy to sketch good direction fields for.

In these cases we resort to numerical methods that will allow us to approximate solutions to differential equations. There are many different methods that can be used to approximate solutions to a differential equation and in fact whole classes can be taught just dealing with the various methods. We are going to look at one of the oldest and easiest to use here. This method was originally devised by Euler and is called Euler's Method.

First, recall an important fact about the IVP  $\frac{dy}{dx} = f(x, y)$ ,  $y(x_0) = y_0$ :

If  $f$  and  $\frac{\partial f}{\partial y}$  are continuous, then the IVP has a unique solution  $y(x)$  in some interval surrounding the initial point  $x = x_0$ .

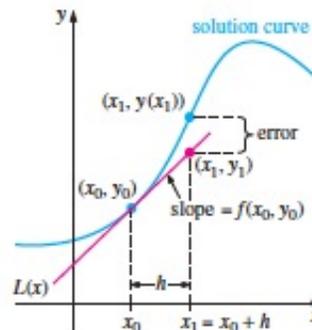
We want to approximate the solution to the IVP near  $x = x_0$ . We'll start with the two pieces of information that we do know about the solution. First, we know the value of the solution at  $x = x_0$  from the initial condition. Second, we also know the value of the derivative at  $x = x_0$ . We can get this by plugging the initial condition into  $f(x, y)$ . So, the derivative at this point is  $f(x_0, y_0)$ .

Now, recall from your calculus class that these two pieces of information are enough for us to write down the equation of the tangent line to the solution at  $x = x_0$ . The tangent line is  $y = L(x)$ , where

$$y = L(x) =$$

We call this the *linearization* of the unknown solution  $y(x)$ .

The graph of  $L(x)$  is a straight line tangent to the graph of  $y = y(x)$  at the point  $(x_0, y_0)$ .



If  $x_1$  is close enough to  $x_0$  then the point  $y_1$  on the tangent line should be fairly close to the actual value of the solution at  $x_1$ , or  $y(x_1)$ . Finding  $y_1$  is easy enough. All we need to do is plug  $x_1$  in the equation for the tangent line.

$$y_1 = L(x_1) =$$

To proceed further, we can try to repeat the process. Unfortunately, we do not know  $y(x_1)$ , the value of the solution at  $x_1$ . The best we can do is use the approximate value  $y_1$  because the point  $(x_1, y_1)$  on the tangent line is an approximation to the point  $(x_1, y(x_1))$  on the solution curve. Now we construct the line through the point  $(x_1, y_1)$  with slope  $f(x_1, y_1)$ .

$$y =$$

To approximate the value of the solution at a nearby point  $x_2$ , we get

$$y_2 =$$

We can continue in this fashion. Use the previously computed approximation to get the next approximation. So,

$$y_3 =$$

$$y_4 =$$

In general, if we have  $x_n$  and the approximation to the solution at this point,  $y_n$ , and we want to find the approximation at  $x_{n+1}$  all we need to do is use the following:

$$y_{n+1} =$$

If we assume that there is a uniform step size  $h$  between the points  $x_0, x_1, x_2, \dots$ , then  $x_{n+1} = x_n + h$ , and we get

$$y_{n+1} =$$

This procedure of using successive “tangent lines” is called **Euler’s method**.

To use Euler’s method, you repeatedly evaluate the last equation, using the result of each step to execute the next step. In this way you generate a sequence of values  $y_1, y_2, y_3, \dots$ , that approximate the values of the solution  $y(x)$  at the points  $x_1, x_2, x_3, \dots$

**Example:** Use Euler’s Method to obtain a four-decimal approximation of the indicated value. First use  $h = 0.1$  and then use  $h = 0.05$ . Find an explicit solution and compare the calculated result with the corresponding values of the exact solution.

$$y' = 2xy, \quad y(1) = 1, \quad y(1.5)$$

The actual values were calculated from the known solution. The **absolute error** is defined to be

$$| \text{actual value} - \text{approximation} |$$

The **relative error** and **percentage error** are

$$\frac{\text{absolute error}}{|\text{actual value}|} \quad \text{and} \quad \frac{\text{absolute error}}{|\text{actual value}|} \times 100$$

$$h = 0.1$$

$x_n$	$y_n$	Actual Value	Abs. Error	% Rel. Error
1.00	1.0000	1.0000	0.0000	0.00
1.10		1.2337		
1.20		1.5527		
1.30		1.9937		
1.40		2.6117		
1.50	2.9278	3.4903	0.5625	16.12

$$h = 0.05$$

$x_n$	$y_n$	Actual Value	Abs. Error	% Rel. Error
1.00	1.0000	1.0000	0.0000	0.00
1.05		1.1079	0.0079	0.72
1.10		1.2337	0.0182	1.47
1.15		1.3806	0.0314	2.27
1.20		1.5527	0.0483	3.11
1.25	1.6849	1.7551	0.0702	4.00
1.30	1.8955	1.9937	0.0982	4.93
1.35	2.1419	2.2762	0.1343	5.90
1.40	2.4311	2.6117	0.1806	6.92
1.45	2.7714	3.0117	0.2403	7.98
1.50	3.1733	3.4903	0.3171	9.08