

## Chapter 7 - Roots, Radicals, and Complex Numbers

### 7.1 Roots and Radicals

#### 7.1.1 Notation and Terminology

In the expression  $\sqrt{x}$  the  $\sqrt{\phantom{x}}$  is called the **radical sign**. The expression under the radical sign is called the **radicand**. Frequently there is a number above the radical, like this:  $\sqrt[n]{x}$ . The number ( $n$  in this case) is called the **index**, this number tells us which root we are taking.

$$\sqrt[n]{x}$$

Such an expression is called a **radical expression**.

**REMARK 1.** When taking the square root, we usually don't write the index, ie we write  $\sqrt{x}$  instead of  $\sqrt[2]{x}$ .

#### 7.1.2 Square Roots

For every **positive** real number,  $a$ , there is a positive square root, written  $\sqrt{a}$

**EX 1.** Consider the number 9, let's find  $\sqrt{9}$ .

**Definition 1.** The **principal square root** of a positive number  $a$ , written  $\sqrt{a}$ , is the positive number  $b$  such that  $b^2 = a$ .

Thus, in this class when we ask for  $\sqrt{9}$  the answer is 3 (positive) since  $3^2 = 9$ .

**REMARK 2.** However, keep in mind that for square roots there is always the principal (positive) square root and the negative square roots

**EX 2.** Find the following:

1.  $\sqrt{49}$

2.  $-\sqrt{49}$

3.  $\sqrt{-49}$

When you square any real number, you get a positive number (for example  $7^2 = 49$  and  $(-7)^2 = 49$ ). Therefore, it is impossible to find a number such that  $b^2 = -49$ . Thus  $\sqrt{-49}$  is not a real number.

### 7.1.3 Cube Roots

The story for cube roots (index equal to three) is a little simpler. For each real number, **there is only one cube root**.

**EX 3.** Find the following cube roots:

$$\sqrt[3]{27}$$

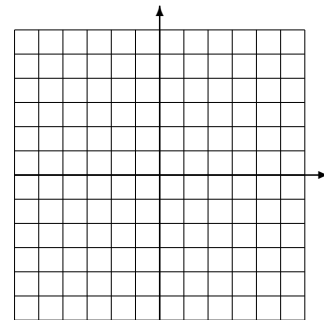
$$\sqrt[3]{-27}$$

### 7.1.4 Radical Functions

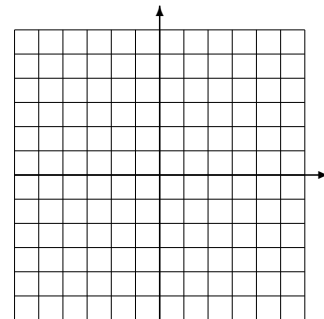
We can use radical expressions to form radical functions:

**EX 4.** -

1. For the function  $f(x) = \sqrt{x}$ , graph by plotting points and find the domain.



2. For the function  $f(x) = \sqrt[3]{x}$ , graph by plotting points and find the domain.



3. For:

(a)  $f(x) = \sqrt{7x + 4}$  find  $f(3)$

(b)  $g(x) = \sqrt{-9x + 1}$  find  $g(1)$

4. For:

(a)  $f(x) = \sqrt[3]{7x-1}$  find  $f(4)$

(b)  $g(x) = \sqrt[3]{13x-1}$  find  $g(-2)$

### 7.1.5 Understand Odd and Even Roots

Recall that the index is the number which indicates which root we are taking. This number can either be even or odd.

- Radicals with odd indices are called **odd roots**:  $\sqrt[3]{a}$ ,  $\sqrt[5]{a}$ ,  $\sqrt[7]{a}$ , ...
  - With an **odd** index any real number can be the radicand and:
    - \* An odd root of a positive number is positive.
    - \* An odd root of a negative number is negative.
- Radicals with even indices are called **even roots**:  $\sqrt{a}$ ,  $\sqrt[4]{a}$ ,  $\sqrt[6]{a}$ , ...
  - With an even index the **radicand must be positive** and the root will be positive.

**EX 5.** Evaluate:

1.  $\sqrt{9}$

2.  $\sqrt[4]{81}$

3.  $\sqrt[6]{729}$

4.  $\sqrt[4]{\frac{1}{81}}$

5.  $\sqrt[3]{27}$

6.  $\sqrt[5]{-243}$

7.  $\sqrt[5]{243}$

### 7.1.6 Evaluate Radicals Using Absolute Values

For any real number  $a$ ,  $\sqrt{a^2} = |a|$ . For example:

$$\begin{aligned}\sqrt{3^2} &= 3 \\ \sqrt{(-3)^2} &= 3\end{aligned}$$

**REMARK 3.** We will only use this method with square roots.

**EX 6.** Simplify:

1.  $\sqrt{25}$

2.  $\sqrt{0}$

3.  $\sqrt{(x+5)^2}$

4.  $\sqrt{a^2 - 8a + 16}$

5. Simplify, assume all variables represent positive values

$$\sqrt{81x^2}$$

## 7.2 Rational Exponents

In this section we learn new notation to write radical expressions as exponential expressions.

### 7.2.1 Changing Radical Expressions to Exponential Expressions

We use the following notation:

$$\sqrt[n]{a} = a^{\frac{1}{n}}$$

**REMARK 4.** • When  $a$  is nonnegative,  $n$  can be any integer.

- When  $a$  is negative,  $n$  must be odd.

**EX 7.** -

1. Write in exponential form:

(a)  $\sqrt{5}$

(b)  $\sqrt[4]{17xy}$

2. Write each expression in radical form:

(a)  $(25)^{\frac{1}{2}}$

(b)  $(12x^3y^2)^{\frac{1}{7}}$

### 7.2.2 Simplify Radical Expressions

We extend this notation further as follows:

$$\sqrt[n]{a^m} = (\sqrt[n]{a})^m = a^{\frac{m}{n}}$$

for any real number  $a$ , and integers  $m$  and  $n$ .

**EX 8.** -

1. Write in exponential form:

(a)  $\sqrt{z^3}$

(b)  $(\sqrt[3]{y})^4$

(c)  $\sqrt[5]{y^2}$

2. Write in exponential form and simplify

(a)  $\sqrt[5]{x^{20}}$

(b)  $(\sqrt[3]{r})^{21}$

3. Write each expression in radical form

(a)  $m^{\frac{3}{5}}$

(b)  $(5ab)^{\frac{7}{4}}$

4. Simplify:

(a)  $36^{\frac{3}{2}}$

(b)  $(\sqrt[20]{z})^4$

**REMARK 5.** What if the index matches the power?

$$\sqrt[n]{a^n} = (\sqrt[n]{a})^n = a^{\sqrt[n]{n}} = a^1 = a$$

### 7.2.3 Rules of Exponents

All the rules we had with exponents before still apply with rational exponents:

For all real numbers  $a$  and  $b$  and all rational numbers  $m$  and  $n$ ,

Product rule	$a^m \cdot a^n = a^{m+n}$
Quotient rule	$\frac{a^m}{a^n} = a^{m-n}, a \neq 0$
Negative exponent rule	$a^{-m} = \frac{1}{a^m}, a \neq 0$
Zero exponent rule	$a^0 = 1, a \neq 0$
Raising a power to a power	$(a^m)^n = a^{m \cdot n}$
Raising a product to a power	$(ab)^m = a^m b^m$
Raising a quotient to a power	$(\frac{a}{b})^m = \frac{a^m}{b^m}, b \neq 0$

**EX 9.** -

1. Evaluate

(a)  $27^{-\frac{2}{3}}$

(b)  $(-32)^{-\frac{4}{5}}$

2. Evaluate:  $(\frac{36}{49})^{-\frac{1}{2}}$

3. Simplify each expression (and write without negative exponents)

(a)  $(5x^4y^{-6})^{\frac{1}{2}}$

(b)  $2x^{\frac{1}{4}}(4x^{\frac{1}{2}} + x^{-\frac{1}{6}})$

(c)  $(\frac{3x^{-2}z^{\frac{3}{5}}}{z^{-\frac{2}{5}}})^{\frac{1}{4}}$

4. Simplify

(a)  $\sqrt[6]{(3y)^4}$

(b)  $(\sqrt[3]{ab^2c^4})^2$

(c)  $\sqrt[3]{\sqrt{x}}$

#### 7.2.4 Factor Expressions with Rational Exponents

To factor expressions with rational exponents, we factor out the term with the smallest (or most negative) exponent.

**EX 10.** Factor  $x^{\frac{3}{7}} + x^{-\frac{4}{7}}$

## 7.3 Simplifying Radicals

In this section we will use the product and quotient rules for radicals to simplify radical expressions. To begin we study perfect powers.

### 7.3.1 Perfect Powers

The simplest perfect power is a **perfect square**, a number or expression which is the square of an expression.

**EX 11.** The following are examples of perfect squares:

**Numbers**

1, 4, 9, 16, 25, ...

**Variables**

$x^2$ ,  $x^4$ ,  $x^6$ ,  $x^8$ ,  $x^{10}$ , ...,

Similarly we have perfect cubes:

**EX 12.** The following are perfect cubes: **Numbers**

1, 8, 27, 64, 125, ...

**Variables**

$x^3$ ,  $x^6$ ,  $x^9$ ,  $x^{12}$ ,  $x^{15}$ , ...,

More generally, perfect powers for an index  $n$ :

**Variables**

$x^n$ ,  $x^{2n}$ ,  $x^{3n}$ ,  $x^{4n}$ ,  $x^{5n}$ , ...,

**REMARK 6.** Note that we can tell if a power is a perfect power for a given  $n$  if the power is divisible by the index. For example:  $\sqrt[7]{n^{35}}$ . The exponent 35 is divisible by the index 7, so  $n^{35}$  is a perfect power for the index 7.

### 7.3.2 Simplifying Using the Product Rule for Radicals

Recall the product rule for exponents: For nonnegative real numbers  $a$  and  $b$ ,

$$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$$

**EX 13.** List all the ways we can factor  $\sqrt{75}$  using the product rule:

**To Simplify Radicals:**

1. For any numbers under the radical (other than 1), write it as a product of two numbers, one of which is the largest perfect power for the index.
2. For each variable factor, write it as a product of two factors, one of which is the largest perfect power of the variable for the index.
3. Use the product rule to separate the perfect powers under one radical sign.
4. Simplify the radical containing the perfect power.

**EX 14.** Simplify the following radicals:

1.  $\sqrt{75}$

2.  $\sqrt[3]{108}$

3.  $\sqrt{x^8}$

4.  $\sqrt{y^{11}}$

5.  $\sqrt[6]{r^{43}}$

**REMARK 7.** After simplifying the radicand we will not have a variable under the radical with an exponent greater than or equal to the index.

6.  $\sqrt{x^{14}y^{21}}$

7.  $\sqrt[4]{x^9y^{27}}$

8.  $\sqrt{50x^7y^{14}z^5}$

9.  $\sqrt[3]{40x^{10}y^{20}}$

### 7.3.3 Simplify Radicals Using the Quotient Rule for Radicals

Recall the quotient rule: For nonnegative real numbers  $a$  and  $b$ ,  $b \neq 0$

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$$

We will frequently use this rule in either direction.

**EX 15.** Simplify.

1.  $\frac{\sqrt{50}}{\sqrt{2}}$

2.  $\frac{\sqrt[3]{x^5y^8}}{\sqrt[3]{x^2y^{-4}}}$

3.  $\sqrt{\frac{169}{49}}$

4.  $\sqrt[3]{\frac{27m^7n^2}{64mn^{17}}}$

**REMARK 8.** Some common mistakes to avoid:

## 7.4 Adding, Subtracting, Multiplying Radicals

### 7.4.1 Add and Subtract Radicals

Adding and subtracting radicals is similar to adding and subtracting variables.

**Definition 2.** **Like radicals** are radicals having the same radicand and index.

**unlike radicals** are radicals differing in either radicand or index.

	<b>like:</b>	<b>unlike</b>
	$\sqrt{3}, 4\sqrt{3}$	$\sqrt{3}, 4\sqrt[3]{3}$
<b>EX 16.</b>	$4\sqrt{7}, -8\sqrt{7}$	$4\sqrt{7}, -8\sqrt{6}$
	$\sqrt[3]{y}, 3\sqrt[3]{y}$	$\sqrt[3]{y}, 3\sqrt{y}$
	$\sqrt[5]{2x}, -8\sqrt[5]{2x}$	$\sqrt[5]{2x}, -8\sqrt[5]{x}$

Like radicals are added in the same manner as like terms, numerical coefficients are added.

**EX 17.** Simplify.

1.  $7 + 5\sqrt{3} - \sqrt{3} + 10$

2.  $4\sqrt[3]{x} + 2x + 5\sqrt[3]{x} - 6$

**REMARK 9.** It is frequently necessary to **simplify** radicals before addition or subtraction can be done.

**Steps:**

(a) Simplify each radical expression.

(b) Combine like radicals

3.  $\sqrt{2} + \sqrt{72}$

4.  $3\sqrt{28} + 2\sqrt{63}$

5.  $2\sqrt{27} + 5\sqrt{48} - 4\sqrt{12}$

6.  $\sqrt[3]{8} + \sqrt[3]{54} - 7\sqrt[3]{2}$

7.  $\sqrt{x^4} - \sqrt{xy^2} + y\sqrt{x}$

8.  $\sqrt[3]{x^{13}y^2} - \sqrt[3]{x^4y^{11}}$

**REMARK 10.** It is tempting but wrong to do the following:

$$\sqrt[n]{a} + \sqrt[n]{b} \neq \sqrt[n]{a+b}$$

### 7.4.2 Multiply Radicals

To multiply radicals we use the product rule.

**EX 18.** Multiply and Simplify.

1.  $\sqrt{3x^5}\sqrt{6x^4}$

2.  $\sqrt[3]{3x^2}\sqrt[3]{9x^4}$

3.  $\sqrt[5]{4x^9y^2}\sqrt[5]{16x^2y^{21}}$

4.  $\sqrt{3x}(\sqrt{27x} - \sqrt{12})$

5.  $(\sqrt{m} + \sqrt{n})(\sqrt{m} - n)$

6.  $(2\sqrt{5} - \sqrt{3})^2$

7.  $(\sqrt[3]{x^2} - \sqrt[3]{3y})(\sqrt[3]{x} + \sqrt[3]{18y^2})$

8.  $(4 + \sqrt{5})(4 - \sqrt{5})$

## 7.5 Dividing Radicals

### 7.5.1 Rationalizing Denominators

Generally we do not leave radicals in the denominators of expressions. When we remove a radical from a denominator it is called **rationalizing the denominator**.

**To Rationalize a Denominator:**

Multiply both the numerator and the denominator of the fraction by a radical that will result in the radicand in the denominator becoming perfect power.

**EX 19.** Simplify.

1.  $\frac{y}{2\sqrt{5}}$

2.  $\frac{\sqrt[3]{32a^5}}{\sqrt[3]{b}}$

3.  $\sqrt{\frac{x^2}{3y}}$

4.  $\sqrt[3]{\frac{x^7y^{10}}{4z}}$

### 7.5.2 Rationalize a Denominator Using the Conjugate

Sometimes an expression will have a binomial in the denominator. To rationalize in this case, we have to multiply by the conjugate.

**Definition 3.** The **conjugate** of a binomial is a binomial having the same two terms with the sign of the second term changed.

	<b>Expression</b>	<b>Conjugate</b>
<b>EX 20.</b>	$4 + \sqrt{3}$	$4 - \sqrt{3}$
	$7\sqrt{3} + \sqrt{2}$	$7\sqrt{3} - \sqrt{2}$
	$\sqrt{m} - \sqrt{n}$	$\sqrt{m} + \sqrt{n}$

Why is this useful? Let's see:

**EX 21.** -

1. Multiply:  $(4 + \sqrt{3})(4 - \sqrt{3})$

2. Simplify  $\frac{3}{3+\sqrt{5}}$

3. Simplify  $\frac{m+\sqrt{n}}{m-\sqrt{n}}$

### 7.5.3 Review of Radical Simplification

How do we know if a radical is simplified?

**A radical is simplified if:**

1. No perfect powers are factors of the radicand and all exponents in the radicand are less than the index.
2. There are no fractions in the radicand.
3. No denominator contains a radical.

**EX 22.** Simplify

1.  $\sqrt{50x^7}$

2.  $\sqrt{\frac{1}{7}}$

3.  $2\sqrt{3} + \frac{4}{\sqrt{3}} + \sqrt{75}$

### 7.5.4 Expressions with Different Indices

We can use the rules of radicals to divide expressions with divide radical expressions where their indices are different.

**EX 23.** Simplify.

1.  $\frac{\sqrt[4]{(m+n)^5}}{\sqrt[3]{(m+n)^2}}$

2.  $\frac{\sqrt[3]{a^4b^5}}{\sqrt{ab^3}}$

## 7.6 Solving Radical Equations

A **radical equation** is an equation that contains a variable in the radicand.

**EX 24.** Some examples of radical equations:

$$\sqrt{y} = 8, \quad \sqrt{x-6} - 3 = 0, \quad \sqrt[3]{x} + 11 = 8$$

**To Solve Radical Equations:**

1. Rewrite the equation so that one radical containing a variable is by itself (isolated) on one side of the equation.
2. Raise each side of the equation to a power equal to the index of the radical.
3. Combine like terms
4. If the equation still contains a term with a variable in a radicand repeat steps 1 through 3.
5. Solve the resulting equation for the variable.
6. Check all solutions in the original equation for extraneous solutions.

**EX 25.** Solve.

1.  $\sqrt{y} = 8$

2.  $\sqrt{x-6} - 3 = 0$

3.  $\sqrt[3]{x} + 11 = 8$

4.  $\sqrt{x} + 5 = 0$

**REMARK 11.** It is necessary to always check your solutions to these equations!

5.  $\sqrt{5x + 4} = x - 4$

6.  $x - 3\sqrt{x} - 4 = 0$

7.  $\sqrt{4x^2 + 5} = 2\sqrt{x^2 + x - 3}$

8. For  $f(x) = 2(x + 5)^{\frac{1}{3}}$  and  $g(x) = (23x - 5)^{\frac{1}{3}}$  find values of  $x$  for which  $f(x) = g(x)$ .

9.  $\sqrt{3x + 1} - \sqrt{2x - 1} = 1$

## 7.7 A Brief Introduction to Complex Numbers

As we saw earlier, negative numbers don't have square roots. This is somehow unsatisfying. To resolve this, we invent a new number system called the **complex numbers**. We define the **imaginary unit**:

$$\sqrt{-1} = i$$

Thus

$$\sqrt{-n} = \sqrt{-1 \cdot n} = \sqrt{-1} \cdot \sqrt{n} = i\sqrt{n}$$

and in particular  $i^2 = -1$ .

**EX 26.** Write as a complex number (with  $i$ ).

1.  $\sqrt{-4}$

2.  $\sqrt{-9}$

More generally, a complex number is a number of the form:

$$a + bi$$

where  $a$  and  $b$  are real numbers and  $i^2 = -1$ .  $a$  is called the real part and  $bi$  is called the imaginary part.

**EX 27.** Complex number arithmetic:

1. Add  $(8 + 5i) + (-6 - i) + 15$

2. Subtract  $(7 - \sqrt{-50}) - (-8 - \sqrt{-72})$

3. Multiply  $2i(9 - 3i)$

4. Multiply  $\sqrt{-16}(\sqrt{-2} + 5)$

5. Multiply  $(2 + \sqrt{-8})(4 - \sqrt{-2})$